Chapter 1 The Real Number System

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1.1 Introduction

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1. 1.2 Ordered Field Axioms

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Postulate 1. [Field Axioms]

There are functions + and $\cdot$ defined on $\mathbb{R}^2 \coloneqq \mathbb{R} \times \mathbb{R}$, which satisfies the following properties for every $a, b, c \in \mathbb{R}$

Closure Properties.

$a + b$ and $a \cdot b$ belong to $\mathbb{R}$

Associative Properties.

$a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Commutative Properties.

$a + b = b + a$ and $a \cdot b = b \cdot a$

Distributive Law.

$a \cdot (b + c) = a \cdot b + a \cdot c$

Existence of the Additive Identity.

There is a unique element $0 \in \mathbb{R}$ such that $0 + a = a$ for all $a \in \mathbb{R}$.

Existence of the Multiplicative Identity.

There is a unique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbb{R}$.

Existence of Additive Inverses.

For every $x \in \mathbb{R}$ there is a unique element $-x \in \mathbb{R}$ such that
$x + (-x) = 0$

Existence of Multiplicative Inverses.

For every $x \in \mathbb{R} / \{0\}$ there is a unique element $x^{-1} \in \mathbb{R}$ such that
$x \cdot (x^{-1}) = 1$

Postulate 2. [Order Axioms].

There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ that has the following properties:

Trichotomy Property.

Given $a, b \in \mathbb{R}$, one and only one of the following statements holds:

$a < b, b < a$, or $a = b$

Transitive Property.

For $a, b, c \in \mathbb{R}$,

$a < b$ and $b < c$ imply $a < c$

The Additive Property.

For $a, b, c \in \mathbb{R}$,

$a < b$ and $c \in \mathbb{R}$ imply $a + c < b + c$.

The Multiplicative Properties.

For $a, b, c \in \mathbb{R}$

$a < b$ and $c > 0$ imply $ac < bc$

And

$a < b$ and $c > 0$ imply $bc < ac$.

By $b > a$ we shall mean $a < b$. By $a \leq b$ and $b \geq a$ we shall mean $a < b$ or $a = b$. By $a < b < c$ we shall mean $a < b$ and $b < c$. In particular, $2 < x < 1$ makes no sense at all.

The real number system $\mathbb{R}$ contains certain special subsets:

The set of natural numbers

$\mathbb{N} \coloneqq \{1,2, \dots\}$

Obtained by beginning with 1 and successively adding 1s to form $2 := 1 + 1.3 := 2 + 1$, and so on;

The set of integers
$\mathbb{Z} := \{\ldots - 2, -1, 0, 1, 2, \ldots\}$
(*Zahlen* is German for number);

The set of rationals (or fractions or quotients)

$$\mathbb{Q} := \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$$

And the set of *irrationals*
$\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$

Equality in $\mathbb{Q}$ is defined by
$\frac{m}{n} = \frac{p}{q}$ if and only if $mq = nq$.

Recall that each of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ is a proper subset of the next; that is

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

1.1 **Remark**
We will assume that the sets of $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties.
1. If $n, m \in \mathbb{Z}$, then $n + m, n - m$, and mn belong to $\mathbb{Z}$.
2. If $n \in \mathbb{Z}$, then $n \in \mathbb{N}$ if and only if $n \geq 1$.
3. There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$.

1.2 **Example**
If $a \in \mathbb{R}$, prove that

$$a \neq 0 \text{ implies } a^2 > 0$$

If particular, $-1 < 0 < 1$.

Proof. Suppose that $a \neq 0$. By the Trichotomy Property, either $a > 0$ or $a < 0$.
Case 1. $a > 0$. Multiply both sides of this inequality by a, using the First Multiplicative Property. We obtain $a^2 = a \cdot a > 0 \cdot a$. Since (by (2)), $0 \cdot a = 0$ we conclude that $a^2 > 0$.
Case 2. $a < 0$. Multiply both sides of this inequality by a. Since $a < 0$, it follows from the Second Multiplicative Property that $a^2 = a \cdot a > 0 \cdot a = 0$. This proves that $a^2 > 0$ when $a \neq 0$.
Since $1 \neq 0$, it follows that $1 = 1^2 > 0$. Adding -1 to both sides of this inequality, we conclude that $0 = 1 - 1 > 0 - 1 = -1$.

1.3 **Example**
If $a \in \mathbb{R}$, prove that

$$0 < a < 1 \text{ implies } 0 < a^2 < a \text{ and } a > 1 \text{ implies } a^2 > a.$$

Proof. Suppose that $0 < a < 1$. Multiply both sides of this inequality by a using the First Multiplicative Property. We obtain $0 = 0 \cdot a < a^2 < 1 \cdot a = a$. In particular, $0 < a^2 < a$.
On the other hand, if $a > 1$, then $a > 0$ by Example 1.2 and the Transitive Property. Multiplying $a > 1$ by a, we conclude that $a^2 = a \cdot a > 1 \cdot a = a$

1.4 **Definition.**
The absolute value of a number $a \in \mathbb{R}$ is the number
$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$

1.5 **Remark**
The absolute value is multiplicative; that is $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$

Proof. We consider four cases.
Case 1. $a = 0$ *or* $b = 0$. Then $ab = 0$, so by definition, $|ab| = 0 = |a||b|$.
Case 2. $a > 0$ and $b > 0$. By the First Multiplicative Property, $ab > 0 \cdot b = 0$. Hence by definition, $|ab| = ab = |a||b|$.

Case 3. $a > 0$ and $b < 0$, or $b > 0$ and $a < 0$. By symmetry, we may suppose that $a > 0$ and $b < 0$. (That is, if we can prove it for $a > 0$ and $b < 0$, then by reversing the roles of a and b, we can prove it for $a < 0$ and $b > 0$.)
By the Second Multiplicative Property, $ab < 0$. Hence by Definition 1.4,(2), and commutativity.
$|ab| = -(ab) = (-1)(ab) = a\big((-1)b\big) = a(-b) = |a||b|$.
Case 4. $a < 0$ and $b < 0$. By the Second Multiplicative Property, $ab > 0$. Hence by Definition 1.4

$$|ab| = ab = (-1)^2(ab) = (-a)(-b) = |a||b|.$$

1.6 **Theorem. [FUNDAMENTAL THEOREM OF ABSOLUTE VALUES]**

Let $a \in \mathbb{R}$ *and* $M \geq 0$. *Then* $|a| \leq M$ *if and only if* $-M \leq a \leq M$

Proof. Suppose first that $|a| \leq M$. Multiplying by -1, we also have $-|a| \geq -M$.
Case 1. $a \geq 0$. By Definition 1.4, $|a| = a$. Thus, by hypothesis,
$-M \leq 0 \leq a = |a| \leq M$
Case 2. $a < 0$. By Definition 1.4, $|a| = -a$. Thus by hypothesis,
$-M \leq -|a| = a < 0 \leq M$
This proves that $-M \leq a \leq M$ in either case.

Conversely, if $-M \leq a \leq M$, then $a \leq M$ and $-M \leq a$. Multiplying the second inequality by -1, we have $-a \leq M$. Consequently, $|a| = a \leq M$ if $a \geq 0$, and $|a| = -a \leq M$ if $a < 0$.

1.7 **Theorem.** The absolute value satisfies the following three properties.

1. [POSITIVE DEFINITE] For all $a \in \mathbb{R}$, $|a| \geq 0$ with $|a| = 0$ if and only if $a = 0$.
2. [SYMMETRIC] For all $a, b \in \mathbb{R}$, $|a - b| = |b - a|$.
3. [TRIANGLE INEQUALITIES] For all $a, b \in \mathbb{R}$,
$$|a + b| \leq |a| + |b| \text{ and } \big||a| - |b|\big| \leq |a - b|$$

Proof.

1. If $a \geq 0$, then $|a| = a \geq 0$. If $a < 0$, then by Definition 1.4 and the Second Multiplicative Property, $|a| = -a = (-1)a > 0$. Thus $|a| \geq 0$ for all $a \in \mathbb{R}$.
   If $|a| = 0$, then by definition $a = |a| = 0$ when $a \geq 0$ and $a = -|a| = 0$ when $a < 0$. Thus $|a| = 0$ implies that $a = 0$. Conversely, $|0| = 0$ by definition.
2. By Remark 1.5, $|a - b| = |-1||b - a| = |b - a|$.
3. To prove the first inequality, notice that $|x| \leq |x|$ holds for any $x \in \mathbb{R}$. Thus Theorem 1.6 implies $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Adding these inequalities (see Exercises 1.2.1), we obtain
   $-(|a| + |b|) \leq a + b \leq |a| + |b|$
   Hence by Theorem 1.6 Again, $|a + b| \leq |a| + |b|$.
   To prove the second inequality, apply the first inequality to $(a - b) + b$. We obtain
   $|a| - |b| = |a - b + b| - |b| \leq |a - b| + |b| - |b| = |a - b|$
   By reversing the roles of a and b and applying part ii), we also obtain
   $|b| - |a| \leq |b - a| = |a - b|$
   Multiplying this last inequality by -1 and combining it with the preceding one verifies
   $-|a - b| \leq |a| - |b| \leq |a - b|$
   We conclude by Theorem 1.6 that $\big||a| - |b|\big| \leq |a - b|$
   Notice once and for all that this last inequality implies that $|a| - |b| \leq |a - b|$ for all $a, b \in \mathbb{R}$. We will use this inequality several times.

1.8 ***Example.***
Prove that if $-2 < x < 1$, then $|x^2 - x| < 6$.
Proof. By hypothesis, $|x| < 2$. Hence by the triangle inequality and Remark 1.5,
$|x^2 - x| \leq |x|^2 + |x| < 4 + 2 = 6$
1.9 **Theorem.**
Let $x, y, a \in \mathbb{R}$.

1. $x < y + \epsilon$ for all $\epsilon > 0$ if and only if $x \leq y$.
2. $x > y - \epsilon$ for all $\epsilon > 0$ if and only if $x \geq y$.
3. $|a| < \epsilon$ for all $\epsilon > 0$ if and only $a = 0$

Proof.

1. Suppose to the contrary that $x < y + \epsilon$ for all $\epsilon > 0$ but $x > y$. Set $\epsilon_0 = x - y > 0$ and

observe that $y+\epsilon_0 = x$. Hence by the Trichotomy Property, $y+\epsilon_0$ cannot be greater than x. This contradicts the hypothesis for $\epsilon = \epsilon_0$. Thus $x \le y$.
Conversely, suppose that $x \le y$ and $\epsilon > 0$ is given. Either $x < y$ or $x = y$. If $x < y$, then $x + 0 < y + 0 < y + \epsilon$ by the Additive and Transitive Properties. If $x = y$, then $x < y + \epsilon$ by Additive Property. Thus $x < y + \epsilon$ for all $\epsilon > 0$ in either case. This completes the proof of part 1.
2. Suppose that $x > y - \epsilon$ for all $\epsilon > 0$. By the Second Multiplicative Property, this is equivalent to $-x < -y + \epsilon$, hence by part 1, equivalent to $-x \le -y$. Multiplying this inequality by -1, we conclude that $x \ge y$.
3. Suppose that $|a| < \epsilon = 0 + \epsilon$ for all $\epsilon > 0$. By part 1, this is equivalent to $|a| \le 0$. Since it is always the case that $|a| \ge 0$, we conclude by the Trichotomy Property that $|a| = 0$. Therefore, $a = 0$ by the Theorem 1.7i

Let a and b be real numbers. A closed interval is a set of the form

$[a,b] \coloneqq \{x \in \mathbb{R}: a \le x \le b\}$.
$[a,\infty) \coloneqq \{x \in \mathbb{R}: a \le x\}$.
$(-\infty,b] \coloneqq \{x \in \mathbb{R}: x \le b\}$,
$(-\infty,\infty) \coloneqq \mathbb{R}$.

And an open interval is a set of the form

$(a,b) \coloneqq \{x \in \mathbb{R}: a < x < b\}$
$(a,\infty) \coloneqq \{x \in \mathbb{R}: a < x\}$.
$(-\infty,b) \coloneqq \{x \in \mathbb{R}: x < b\}$.
$(-\infty,\infty) \coloneqq \mathbb{R}$.

By an interval we mean a closed interval, an open interval, or a set of the form

$[a,b) \coloneqq \{x \in \mathbb{R}: a \le x < b\}$ Or
$(a,b] \coloneqq \{x \in \mathbb{R}: a < x \le b\}$

Notice, then, that when $a < b$, the intervals $[a,b]$, $[a,b)$, $(a,b]$, and (a,b) correspond to line segments on the real line, but when $b < a$, these "intervals" are all the empty set.

An interval *I* is said to be bounded if and only if it has the form $[a,b]$, (a,b), $[a,b)$, or $(a,b]$ for some $-\infty < a \le b < \infty$, in which case the numbers a, b will be called the endpoints of I. All other intervals will be called unbounded. An interval with endpoints a, b is called degenerate if $a = b$ and nondegenerate if $a < b$. Thus a degenerate open interval is the empty set, and a degenerate closed interval is a point.

Exercises

1.2.0
Let $a, b, c, d \in \mathbb{R}$ and consider each of the following statements. Decide which are true and which are false. Prove the true ones and give counterexamples to the false ones.
1. If $a < b$ and $c < d < 0$, then $ac < bd$.
   False.
   $1 < 2$ And $-2 < -1 < 0$
   But
   $-2 \not< -2$
2. If $a \le b$ and $c > 1$, then $|a + c| \le |b + c|$.
   False

   $-6 \le -5$
   $2 > 1$
   $|-6+2| \not\le |-5+2|$

3. If $a \leq b$ and $b \leq a + c$, then $|a - b| \leq c$.
   True
   By the Field Axioms, there exist $-b$ such that
   $b + (-b) = 0$
   By Additive Property
   $a + (-b) \leq b + (-b) = 0$
   Thus,
   $a - b \leq 0$
   $|a - b| = b - a$
   Since we know that
   $b \leq a + c$
   $b - a \leq c$
   $|a - b| \leq c$

   Thus, the statement is true.
4. If $a < b - \epsilon$ for all $\epsilon > 0$, then $a < 0$.
   True????

   Not sure how to prove it
   But it seems that in order to let the hypothesis to be true, $a = -\infty, b = \infty$

   By Multiplicative Properties
   $-a > (-b) + \epsilon$ for all $\epsilon > 0$

1.2.1

Suppose that $a, b, c \in \mathbb{R}$ and $a \leq b$.

1. Prove that $a + c \leq b + c$.
   By Additive Property (I know that it is not the precise Additive Property, but it is a fairly straightforward Proof so …)
   $a + c \leq b + c$
2. If $c \geq 0$, prove that $a \cdot c \leq b \cdot c$.
   We first invoke the Trichotomy Property to break the inequality down, then
   By the Multiplicative Properties
   When $c = 0$, the equation is equal.

1.2.2.

Prove (7), (8), and (9). Show that each of these statements is false if the hypothesis $a \geq 0$ or $a > 0$ is removed.

1.2.3 This exercise is used in Section 6.3

The positive part of an $a \in \mathbb{R}$ is defined by

$$a^+ := \frac{|a| + a}{2}$$

And the negative part by

$$a^- := \frac{|a| - a}{2}$$

1. Prove that $a = a^+ - a^-$ and $|a| = a^+ + a^-$

$$a^+ - a^- = \frac{|a| + a}{2} - \frac{|a| - a}{2} = \frac{2a}{2} = a$$

$$a^+ + a^- = \frac{|a| + a}{2} + \frac{|a| - a}{2} = \frac{2|a|}{2} = |a|$$

2. Prove that

$$a^+ = \begin{cases} a & a \geq 0 \\ 0 & a \leq 0 \end{cases} \text{ and } a^- = \begin{cases} 0 & a \geq 0 \\ -a & a \leq 0 \end{cases}$$

   It is an easy argument by simply expand the absolute value. So I skip this question.

1.2.4.

Solve each of the following inequalities for $x \in \mathbb{R}$

1. $|2x+1| < 7$

   $-7 < 2x+1 < 7$

   $-8 < 2x < 6$

   $-4 < x < 3$

2. $|2-x| < 2$

   $-2 < (2-x) < 2$

   $-4 < -x < 0$

   $0 < x < 4$

3. $|x^3 - 3x + 1| < x^3$

   $-x^3 < x^3 - 3x + 1 < x^3$

   $-2x^3 < -3x + 1 < 0$

   $-2x^3 + 3x - 1 < 0 < 3x - 1$

   It breaks down to two inequalities

   $-2x^3 + 3x - 1 < 0 \ \Rightarrow 2x^3 - 3x + 1 > 0 \Rightarrow \frac{1}{2}\left(-1-\sqrt{3}\right) < x < \frac{1}{2}\left(\sqrt{3}-1\right) \cup x > 1$

   $3x - 1 > 0 \quad \Rightarrow 3x > 1 \Rightarrow x > \frac{1}{3}$

4. $\frac{x}{x-1} < 1$

   Times both side by $(x-1)^2$

   Since it s a non-negative number

   We can use Multiplicative Properties

   $x(x-1) < (x-1)^2$

   Expand it we get

   $x^2 - x < x^2 - 2x + 1$

   And we apply the Additive Property

   $-x < -2x + 1$

   $0 < -x + 1$

   $x < 1$

5. $\frac{x^2}{4x^2-1} < \frac{1}{4}$

   Skip for the sake of my mind

1.2.5.

Let $a, b \in \mathbb{R}$

1. Prove that if $a > 2$ and $b = 1 + \sqrt{a-1}$, then $2 < b < a$.

   $a - 1 > 1$

   $(a-1)^2 > 1$

   $0 < a - 1 < a$

   $0 < \sqrt{a-1} < \sqrt{a}$

   $1 < \sqrt{a-1} + 1 < \sqrt{a} + 1$

   $\sqrt{a-1} > 1$

   $1 + \sqrt{a-1} > 2$

   $b - a = 1 + \sqrt{a-1} - a$

   $1 < \sqrt{a-1} < a - 1$

   $2 < \sqrt{a-1} + 1 < \sqrt{2} + 1$

   $1 < a - 1 < a$

   $1 < \sqrt{a-1} < \sqrt{a} < a$

2. Prove that if $2 < a < 3$ and $b = 2 + \sqrt{a-2}$, then $0 < a < b$.

   $0 < a - 2 < 1$

3. Prove that if $0 < a < 1$ and $b = 1 - \sqrt{1-a}$, then $0 < b < a$.
4. Prove that if $3 < a < 5$ and $b = 2 + \sqrt{a-2}$, then $3 < b < a$.

1.2.10.

Prove that $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$

$a^2b^2 + 2abcd + c^2d^2$

$a^2b^2 + a^2d^2 + c^2b^2 + c^2d^2$

$2abcd = 2(ad)(cb)$

$a^2d^2 + c^2b^2$

$(ad)^2 + (cb)^2$

$(ad - cb)^2 \geq 0$
$(ad)^2 + (cb)^2 \geq 2abcd$

1.2.11.

1. Let $\mathbb{R}^+$ represent the collection of positive real numbers. Prove that $\mathbb{R}^+$ satisfies the following two properties.
   a. For each $x \in \mathbb{R}$, one any only one of the following holds:
      $x \in \mathbb{R}^+, -x \in \mathbb{R}^+$, or $x = 0$

      Because of Trichotomy Property, that
      $x, 0 \in \mathbb{R}$
      One and only one of the following statements holds:
      $x < 0, x > 0,$ or $x = 0$

   b. Given $x, y \in \mathbb{R}^+$, both $x + y$ and $x \cdot y$ belong to $\mathbb{R}^+$.
2. Suppose that $\mathbb{R}$ contains a subset $\mathbb{R}^+$ (not necessarily the set of positive numbers) which satisfies properties 1 and 2. Define $x \prec y$ by $y - x \in \mathbb{R}^+$. Prove that Postulate 2 holds with $\prec$ in place of $<$.

1.3 Completeness Axiom

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1.10 Definition
Let $E \subset \mathbb{R}$ be nonempty.

1. The set E is said to be *bounded above* if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$, in which case M is called an *upper bound* of E
2. A number s is called a *supremum* of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case we shall say that E has *a finite supremum* s and write $s = \sup E$.)

1.11 *EXAMPLE.*
If $E = [0,1]$, prove that $\sup E = 1$.
Proof.
By the definition of interval, 1 is an upper bound of E. Let M be any upper bound of E; that is, $M \geq x$ for all $x \in E$. Since $1 \in E$, it follows that $M \geq 1$. Thus 1 is the smallest upper bound of E.

1.12 Remark.
If a set has one upper bound, it has infinitely many upper bounds.
Proof. *If* M_0 is an upper bound for a set E, then so is M for any $M > M_0$.

1.13 Remark.
If a set has a supremum, then it has only one supremum.
Proof. Let s_1 and s_2 be suprema of the same set E. Then both s_1 and s_2 are upper bounds of E, whence by Definition 1.10ii, $s_1 \leq s_2$ and $s_2 \leq s_1$. We conclude by the Trichotomy Property that $s_1 = s_2$.

NOTE: This proof illustrates a general principle. When asked to prove $a = b$, it is often easier to verify that $a \leq b$ and $b \leq a$ separately.

1.14 Theorem [Approximation Property for Suprema].
If E has a finite supremum and $\epsilon > 0$ is any positive number, then there is a point $a \in E$ such that
$\sup E - \epsilon < a \leq \sup E$
Proof.
Suppose that the theorem is false. Then there is an $\epsilon_0 > 0$ such that no element of E lies between $s_0 \coloneqq \sup E - \epsilon_0$ and $\sup E$. Since $\sup E$ is an upper bound for E, it follows that $a \leq s_0$ for all $a \in E$; that is s_0 is an upper bound of E. Thus, by Definition 1.10ii, $\sup E \leq s_0 = \sup E - \epsilon_0$. Adding $\epsilon_0 - \sup E$ to both sides of this inequality, we conclude that $\epsilon_0 \leq 0$, a contradiction.

1.15 Theorem.
If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in E$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.
Proof.
Suppose that $s \coloneqq \sup E$ and apply the Approximation Property to choose an $x_0 \in E$ such that $s - 1 < x_0 \leq s$. If $s = x_0$, then $s \in E$. Otherwise, $s - 1 < x_0 < s$ and we can apply the Approximation Property again to choose $x_1 \in E$ such that $x_0 < x_1 < s$.
Subtract x_0 from this last inequation to obtain $0 < x_1 - x_0 < s - x_0$. Since $-x_0 < 1 - s$, it follows that $0 < x_1 - x_0 < s + (1 - s) = 1$. Thus $x_1 - x_0 \in \mathbb{Z} \cap (0,1)$, a contradiction by Remark 1.1iii. We conclude that $s \in E$.

Postulate 3. [Completeness Axiom].

If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

1.16 Theorem [Archimedean Principle]
Given real numbers a and b, with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

Strategy: The idea behind the proof is simple. By the completeness Axiom and Theorem 1.15, any nonempty subset of integers that is bounded above has a "largest" integer. If k_0 is the largest integer that satisfies $k_0 a \leq b$, then $n = (k_0 + 1)$ (which is larger than k_0) must satisfy $na > b$. In order to justify this application of the Completeness Axiom, we have two details to attend to: (1) Is the set $E \coloneqq \{k \in \mathbb{N}: ka \leq b\}$ bounded above? (2) Is E nonempty? The answer to the second question depends on whether $b < a$ or not. Here are the details.

Proof.
If $b < a$, set $n = 1$. If $a \leq b$, consider the set $E = \{k \in \mathbb{N}: ka \leq b\}$. E is nonempty since $1 \in E$. Let $k \in E$ (i.e., $ka \leq b$). Since $a > 0$, it follows from the First Multiplicative Property that $k \leq \frac{b}{a}$. This proves that E is bounded above by $\frac{b}{a}$. Thus by the Completeness Axiom and Theorem 1.15, E has a finite supremum s that belongs to E, in particular, $s \in \mathbb{N}$.
Set $n = s + 1$. Then $n \in \mathbb{N}$ and (since n is larger than s), n cannot belong to E. Thus $na > b$.

1.17 EXAMPLE
Let $A = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$ and $B = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$. Prove that $\sup A = \sup B = 1$.
Proof.
It is clear that 1 is an upper bound of both sets. It remains to see that 1 is the smallest upper bound of both sets. For A, this is trivial. Indeed, if M is any upper bound of A, then $M \geq 1$ (since $1 \in A$). On the other hand, if M is an upper bound for B, but $M < 1$, then $1 - M > 0$. In particular, $\frac{1}{1-M} \in \mathbb{R}$.
Choose, by the Archimedean Principle, an $n \in \mathbb{N}$ such that $n > \frac{1}{1-M}$. It follows (do the algebra) that $x_0 \coloneqq 1 - \frac{1}{n} > M$. Since $x_0 \in B$, this contradicts the assumption that M is an upper bound of B (see Figure 1.3)

1.18 Theorem [Density of Rationals].
If $a, b \in \mathbb{R}$ satisfy $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < q < b$.

Strategy: To find a fraction $q = \frac{m}{n}$ such that $a < q < b$, we must specify both numerator m and denominator n. Let's suppose first that $a > 0$ and that the set $E \coloneqq \left\{k \in \mathbb{N}: \frac{k}{n} \leq a\right\}$ has a supremum, k_0. Then $m \coloneqq k_0 + 1$, being greater than the supremum of E, cannot belong to E. Thus $\frac{m}{n} > a$. Is this the fraction we look for? Is $\frac{m}{n} < b$? Not unless n is large enough. To see this, look at a concrete example: $a = \frac{2}{3}$ and $b = 1$. If $n = 1$, then E has no supremum. When $n = 2$, $k_0 = 1$ and when $n = 3$, $k_0 = 2$. In both cases $\frac{k_0+1}{n} = 1$ is too big. However, when $n = 4$, $k_0 = 2$ so $\frac{k_0+1}{4} = \frac{3}{4}$ is smaller than b, as required.
How can we prove that for each fixed $a < b$ there always is an n larger enough so that if k_0 is chosen as above, then $\frac{k_0+1}{n} < b$? By the choice of k_0, $\frac{k_0}{n} \leq a$. Let's look at the worst case scenario: $a = \frac{k_0}{n}$. Then $b > \frac{k_0+1}{n}$ means

$$b > \frac{k_0 + 1}{n} = \frac{k_0}{n} + \frac{1}{n} = a + \frac{1}{n}$$

(i.e., $b - a > \frac{1}{n}$). Such an n can always be chosen by the Archimedean Principle.
What about the assumption that sup E exists? This requires that E be nonempty and bounded above. Once n is fixed, E will be bounded abouve by na. But the only way that E is nonempty is that at the very least, $1 \in E$ (i.e., that $\frac{1}{n} \leq a$). This requires a second restriction on n. We begin our formal proof at this point.

Proof.

Suppose first that $a > 0$. Since $b - a > 0$, use the Archimedean Principle to choose an $n \in \mathbb{N}$ that satisfies

$$n > \max\left\{\frac{1}{a}, \frac{1}{b-a}\right\},$$

And observe that both $\frac{1}{n} < a$ and $\frac{1}{n} < b - a$.

Consider the set $E = \left\{k \in \mathbb{N}: \frac{k}{n} \le a\right\}$. Since $1 \in E$, E is nonempty. Since $n > 0$, E is bounded above by na. Hence, by Theorem 1.15, $k_0 \coloneqq \sup E$ exists and belong to E, in particular, to $\mathbb{N}$. Set $m = k_0 + 1$ and $q = \frac{m}{n}$. Since k_0 is the supremum of E, $m \notin E$. Thus $q > a$. On the other hand, since $k_0 \in E$, it follows from the choice of n that

$$b = a + (b - a) \ge \frac{k_0}{n} + (b - a) > \frac{k_0}{n} + \frac{1}{n} = \frac{m}{n} = q.$$

Now suppose that $a \le 0$. Choose, by the Archimedean Principle, an integer $k \in \mathbb{N}$ such that $k > -a$. Then $0 < k + a < k + b$, and by the case already proved, there is an $r \in \mathbb{Q}$ such that $k + a < r < k + b$. Therefore, $q \coloneqq r - k$ belongs to $\mathbb{Q}$ and satisfies the inequality $a < q < b$.

1.19 Definition.

Let $E \subset \mathbb{R}$ be nonempty.

1. The set E is said to be bounded below if and only if there is an $m \in \mathbb{R}$ such that $a \ge m$ for all $a \in E$, in which case m is called a lower boudn of the set E.
2. A number t is called an infimum of the set E if and only if t is a lower bound of E and $t \ge m$ for all lower bounds m of E. In this case we shall say that E has an infumum t and write $t = \inf E$.
3. E is said to be bounded if and only if it is bounded both above and below.

1.20 Theorem. [Reflection Principle].

Let $E \subseteq \mathbb{R}$ be nonempty.

1. E has a supremum if and only if $-E$ has an infimum, in which case $\inf(-E) = -\sup E$.
2. E has an infimum if and only if $-E$ has a supremum, in which case $\sup(-E) = -\inf E$.

Proof.

The proofs of these statements are similar. We prove only the first statement.

Suppose that E has a supremum s and set $t = -s$. Since s is an upper bound for E, $s \ge a$ for all $a \in E$, so $-s \le -a$ for all $a \in E$. Therefore, t is a lower bound of $-E$. Suppose that m is any lower bound of $-E$. Then $m \le -a$ for all $a \in E$, so $-m$ is an upper bound of E. Since s is the supremum of E, it follows that $s \le -m$ (i.e., $t = -s \ge m$). Thus t is the infimum of $-E$ and $\sup E = s = -t = -\inf(-E)$.

Conversely, suppose that $-E$ has an infimum t. By definition, $t \le -a$ for all $a \in E$. Thus $-t$ is an upper bound for E. Since E is nonempty, E has a supremum by the Completeness Axiom.

1.21 Theorem [Monotone Property].

Suppose that $A \subseteq B$ are nonempty subsets of $\mathbb{R}$.

1. If B has a supremum, then $\sup A \le \sup B$.
2. If B has an infimum, then $\inf A \ge \inf B$.

Proof.

1. Since $A \subseteq B$, and upper bound of B is an upper bound of A. Therefore, $\sup B$ is an upper bound of A. It follows from the Completeness Axiom that $\sup A$ exists, and from Definition 1.10ii that $\sup A \le \sup B$.
2. Clearly, $-A \subseteq -B$. Thus by part i), Theorem 1.20, and the Second Multiplicative Property, $\inf A = -\sup(-A) \ge -\sup(-B) = \inf B$.

It is convenient to extend the definition of suprema and infima to all subsets of $\mathbb{R}$. To do this we expand the definition of $\mathbb{R}$ as follows. The set of extended real numbers is defined to be $\overline{\mathbb{R}} \coloneqq \mathbb{R} \cup \{\pm\infty\}$. Thus x is an extended real number if and only if either $x \in \mathbb{R}$, $x = +\infty$, or $x = -\infty$.

Let $E \subseteq \mathbb{R}$ be nonempty. We shall define $\sup E = +\infty$ if E is unbounded above and $\inf E = -\infty$ if E is unbounded below. Finally, we define $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. Notice, then, that the supremum of a subset E of $\mathbb{R}$ (respectively, the infimum of E) is finite if and only if E is nonempty and bounded above (respectively, nonempty and bounded below).

Exercise

1.30 Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

1. If A and B are nonempty, bounded subsets of $\mathbb{R}$, then $\sup(A \cap B) \leq \sup A$.

   Since

1. Let ϵ be a positive real number. If A is a nonempty, bounded subset of $\mathbb{R}$ and $B = \{\epsilon x : x \in A\}$, then $\sup(B) = \epsilon \sup(A)$.
2. If $A + B \coloneqq \{a + b : a \in A \text{ and } b \in B\}$, where A and B are nonempty, bounded subsets of $\mathbb{R}$, then $\sup(A + B) = \sup(A) + \sup(B)$.
3. If $A - B \coloneqq \{a - b : a \in A \text{ and } b \in B\}$, where A and B are nonempty, bounded subsets of $\mathbb{R}$, then $\sup(A - B) = \sup(A) - \sup(B)$.

1.4 Mathematical Induction

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1.22 Theorem. [Well-Ordering Principle].
If E is a nonempty subset of $\mathbb{N}$, then E has a least element (i.e., E has a finite infimum and $\inf E \in E$).
Proof.
Suppose that $E \subseteq \mathbb{N}$ is nonempty. Then $-E$ is bounded above, by -1, so by the Completeness Axiom $\sup(-E)$ exists, and by Theorem 1.15, $\sup(-E) \in -E$. Hence by Theorem 1.20, $\inf E = -\sup(-E)$ exists, and $\inf E \in -(-E) = E$.

1.23 Theorem.
Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition (i.e., a verbal statement or formula) which satisfies the following two properties:

1. $A(1)$ is true.
2. For every $n \in \mathbb{N}$ for which $A(n)$ is true, $A(n+1)$ is also true.

Then $A(n)$ is true for all $n \in \mathbb{N}$.
Proof.
Suppose that the theorem is false. Then the set $E = \{n \in \mathbb{N}: A(n) \text{ is false}\}$ is nonempty. Hence by the Well-Ordering Principle, E has a least element, say x.
Since $x \in E \subseteq \mathbb{N} \subset \mathbb{Z}$, we have by Remark 1.1ii that $x \geq 1$. Since $x \in E$, we have by hypothesis 1 that $x \neq 1$. In particular, $x - 1 > 0$. Hence, by Remark 1.1i and iii, $x - 1 \geq 1$ and $x - 1 \in \mathbb{N}$.
Since $x - 1 < x$ and x is a least element of E, the statement $A(x-1)$ must be true. Applying hypothesis ii) to $n = x - 1$, we see that $A(x) = A(n+1)$ must also be true; that is, $x \notin E$, a contradiction.

1.24 *EXAMPLE.*
Prove that
$$\sum_{k=1}^{n}(3k-1)(3k+2) = 3n^3 + 6n^2 + n$$
For $n \in \mathbb{N}$.
Proof.
Let $A(n)$ represent the statement
$$\sum_{k=1}^{n}(3k-1)(3k+2) = 3n^3 + 6n^2 + n.$$
For $n = 1$ the left side of this equation is $2 \cdot 5$ and the right side is $3 + 6 + 1$. Therefore, $A(1)$ is true. Suppose that $A(n)$ is true for some $n \geq 1$. Then
$$\begin{aligned}\sum_{k=1}^{n+1}(3k-1)(3k+2) &= (3n+2)(3n+5) + \sum_{k=1}^{n}(3k-1)(3k+2)\\ &= (3n+2)(3n+5) + 3n^3 + 6n^2 + n\\ &= 3n^3 + 15n^2 + 22n + 10\end{aligned}$$
On the other hand, a direct calculation reveals that
$3(n+1)^3 + 6(n+1)^2 + (n+1) = 3n^3 + 15n^2 + 22n + 10$
Therefore, $A(n+1)$ is true when $A(n)$ is. We conclude by induction that $A(n)$ holds for all $n \in \mathbb{N}$.

1.25 *Lemma.*
If $n, k \in \mathbb{N}$ and $1 \leq k \leq n$, then
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof.
By definition,

$$\binom{n}{k-1}+\binom{n}{k}=\frac{n!\,k}{(n-k+1)!\,k!}+\frac{n!\,(n-k+1)}{(n-k+1)!\,k!}$$
$$=\frac{n!\,(n+1)}{(n-k+1)!\,k!}=\binom{n+1}{k}$$

1.26 Theorem. [Binomial Formula].
If $a, b \in \mathbb{R}, n \in \mathbb{N}$, and 0^0 is interpreted to be 1, then
$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$
Proof.
The proof is by induction on n. The formula is obvious for $n = 1$. Suppose that the formula is true for some $n \in \mathbb{N}$. Then by the inductive hypothesis and Postulate 1,

$$(a+b)^{n+1}=(a+b)(a+b)^n$$
$$=(a+b)\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\right)$$
$$=\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k+1}b^k\right)+\left(\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1}\right)$$
$$=\left(a^{n+1}+\sum_{k=1}^{n}\binom{n}{k}a^{n-k+1}b^k\right)+\left(b^{n+1}+\sum_{k=0}^{n-1}\binom{n}{k}a^{n-k}b^{k+1}\right)$$
$$=a^{n+1}+\sum_{k=1}^{n}\left(\binom{n}{k}+\binom{n}{k-1}\right)a^{n-k+1}b^k+b^{n+1}$$
Hence it follows from Lemma 1.25 that
$$(a+b)^{n+1}=a^{n+1}+\sum_{k=1}^{n}\binom{n+1}{k}a^{n+1-k}b^k+b^{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k;$$
That is, the formula is true for $n + 1$. We conclude by induction that the formula holds for all $n \in \mathbb{N}$.

1.27 Remark.
If $x > 1$ and $x \notin \mathbb{N}$, then there is an $n \in \mathbb{N}$ such that $n < x < n + 1$.

Proof.
By the Archimedean Principle, the set $E = \{m \in \mathbb{N}: x < m\}$ is nonempty. Hence by the Well-Ordering Principle, E has a least element, say m_0.
Set $n = m_0 - 1$. Since $m_0 \in E, n + 1 = m_0 > x$. Since m_0 is least, $n = m_0 - 1 \le x$. Since $x \notin \mathbb{N}$, we also have $n \neq x$. Therefore, $n < x < n + 1$.

1.28 Remark.
If $n \in \mathbb{N}$ is not a perfect square (i.e., if there is no $m \in \mathbb{N}$ such that $n = m^2$), then $\sqrt{n}$ is irrational.
Proof.
Suppose to the contrary that $n \in \mathbb{N}$ is not a perfect square but $\sqrt{n} \in \mathbb{Q}$; that is, $\sqrt{n} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$. Choose by Remark 1.27 an integer $m_0 \in \mathbb{N}$ such that
$m_0 < \sqrt{n} < m_0 + 1$.
Consider the set $E \coloneqq \{k \in \mathbb{N}: k\sqrt{n} \in \mathbb{Z}\}$. Since $q\sqrt{n} = p$, we know that E is nonempty. Thus by the Well-Ordering Principle, E has a least element, say n_0.
Set $x = n_0(\sqrt{n} - m_0)$. By (10), $0 < \sqrt{n} - m_0 < 1$. Multiplying this inequality by n_0, we find that $0 < x < n_0$.
Since n_0 is a least element of E, it follows from (11) that $x \notin E$. On the other hand,
$x\sqrt{n} = n_0(\sqrt{n} - m_0)\sqrt{n} = n_0 n - m_0 n_0\sqrt{n} \in \mathbb{Z}$
Since $n_0 \in E$. Moreover, since $x > 0$ and $x = n_0\sqrt{n} - n_0 m_0$ is the difference of two integers, $x \in \mathbb{N}$. Thus $x \in E$, a contradiction.

1.5 Inverse Functions and Images

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Let X and Y be sets and $f: X \to Y$.

1. f is said to be 1-1 (one to one or an injection) if and only if
   $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$ imply $x_1 = x_2$.
2. f is said to be onto (or a surjection) if and only if for each $y \in Y$ there is a $x \in X$ such that $y = f(x)$.
3. f is called a bijection if and only if it is both 1-1 and onto.
   Sometimes, to emphasize the domain and range of f, we shall say that a bijection $f: X \to Y$ is 1-1 from X onto Y.

1.30 Theorem.

Let X and Y be sets and $f: X \to Y$. Then the following three statements are equivalent.

1. f has an inverse;
2. f is 1-1 from X onto Y;
3. There is a function $g: Y \to X$ such that

$g(f(x)) = x$ for all $x \in X$	(13)

   And

$f\left(g(y)\right) = y$ for all $y \in Y$.	(14)

   Moreover, for each $f: X \to Y$, there is only one function g that satisfies (13) and (14). It is the inverse function f^{-1}.

Proof.

1. implies 2. By definition, if f has an inverse, then $\text{Ran}(f) = Y$ (so f takes X onto Y) and each $y \in Y$ has a unique preimage in X [so, if $f(y_1) = f(y_2)$, then $y_1 = y_2$, i.e., f is 1-1 on X].

2. implies 3. The proof that 1. implies 2. also shows that if $f: X \to Y$ is 1-1 and onto, then f has an inverse. In particular, $g(y) \coloneqq f^{-1}(y)$ satisfies (13) and (14) by (12)

3. implies 1. Suppose that there is a function $g: Y \to X$ which satisfies (13) and (14). If some $y \in Y$ has two preimages, say $x_1 \neq x_2$ in X, then $f(x_1) = y = f(x_2)$. It follows from (13) that $x_1 = g\left(f(x_1)\right) = g\left(f(x_2)\right) = x_2$, a contradiction. On the other hand, given $y \in Y$, set $x = g(y)$. Then $f(x) = f\left(g(y)\right) = y$ by (14), so $\text{Ran}(f) = Y$.

Finally, suppose that h is another function which satisfies (13) and (14), and fix $y \in Y$. By 2., there is an $x \in X$ such that $f(x) = y$. Hence by (13)

$h(y) = h(f(x)) = x = g(f(x)) = g(y)$;

That is $h = g$ on Y. It follows that the function g is unique.

1.31 Remark.

Let I be an interval and let $f: I \to \mathbb{R}$. If the derivative of f is either always positive on I, or always negative on I, then f is 1-1 on I.

Proof.

By symmetry, we may suppose that the derivative f' of f satisfies $f'(x) > 0$ for all $x \in I$. We will use a result that almost everyone who has studied one variable calculus remembers (for a proof, see Theorem 4.17): If $f' > 0$ on an interval I, then f is strictly increasing on I; that is, $x_1, x_2 \in I$ and $x_1 < x_2$ imply that $f(x_1) < f(x_2)$.

To see why this implies that f is 1-1, suppose that $f(x_1) = f(x_2)$ for some x_1, x_2 in X. If $x_1 \neq x_2$, then it follows from the trichotomy property that either $x_1 < x_2$ or $x_2 < x_1$. Since f is strictly increasing on I, either $f(x_1) < f(x_2)$ or $f(x_2) < f(x_1)$. Both of these conclusions contradict the assumption that $f(x_1) = f(x_2)$.

By Theorem 1.30, $f: X \to Y$ has an inverse function f^{-1} if and only if $f^{-1}(f(x)) = x$ for all $x \in X$ and $f\left(f^{-1}(y)\right) = y$ for all $y \in Y$. This suggests that we can find a formula for f^{-1} if $y = f(x)$

can be solved for x.

1.32 Example.
Prove that $f(x) = e^x - e^{-x}$ is 1-1 on $\mathbb{R}$ and find a formula for f^{-1} on $\text{Ran}(f)$.

Solution.
Since $f'(x) = e^x + e^{-x} > 0$ for all $x \in \mathbb{R}$, f is 1-1 on $\mathbb{R}$ by Remark 1.31.
Let $y = e^x - e^{-x}$. Multiplying this equation by e^x and collecting all non-zero terms on one side of the equation, we have
$e^{2x} - ye^x - 1 = 0,$
A quadratic in e^x. By the quadratic formula,

$$e^x = \frac{\left(y \pm \sqrt{y^2+4}\right)}{2}$$

Since e^x is always positive, the minus sign must be discarded. Taking the logarithm of this last identity, we obtain $x = \log\left(y + \sqrt{y^2+4}\right) - \log 2$. Therefore,

$$f^{-1}(x) = \log\left(x + \sqrt{x^2+4}\right) - \log 2 .$$

1.33 Definition
Let X and Y be sets and $f: X \to Y$. The image of a set $E \subseteq X$ under f is the set
$f(E) := \{y \in Y: y = f(x) \text{ for some } x \in E\}$
The inverse image of a set $E \subseteq Y$ under f is the set
$f^{-1}(E) := \{x \in X: f(x) = y \text{ for some } y \in E\}$

1.34 Example
Find the images and inverse images of the sets $I = (-1,0)$ and $J = (0,1]$ under the function $f(x) = x^2 + x$.
Solution:
Since "find" doesn't mean "prove", we look at the graph $y = x^2 + x$. By definition, $f(I)$ consists of the y-values of $f(x)$ as x ranges over $I = (-1, 0)$.
Since f has roots at $x = 0, -1$ and has minimum of -0.25 at $x = -0.5$, it is clear by looking at the graph that $f(I) = [-0.25, 0)$. Since $f^{-1}(I)$ consist of the x-values whose images belong to $I = (-1,0)$, and the graph of f lies below the x-axis only when $-1 < x < 0$, it is also clear that $f^{-1}(I) = (-1, 0)$. Similarly, $f(J) = (0, 2]$ and

$$f^{-1}(J) = \left[\frac{\left(-1-\sqrt{5}\right)}{2}, 1\right) \cup \left(0, \frac{-1+\sqrt{5}}{2}\right]$$

1.35 Definition.
Let $\mathcal{E} = \{E_\alpha\}_{\alpha \in A}$ be a collection of sets
1. The union of the collection $\mathcal{E}$ is the set
$$\bigcup_{\alpha \in A} E_\alpha := \{x: x \in E_\alpha \text{ for some } \alpha \in A\}.$$
2. The intersection of the collection $\mathcal{E}$ is the set
$$\bigcap_{\alpha \in A} E_\alpha := \{x : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

1.36 Theorem. [Demorgan's Laws].
Let X be a set and $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of X. If for each $E \subseteq X$ the symbol E^c represents the set $X \setminus E$, then

$$\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c \quad (17)$$

And

$$\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c. \quad (18)$$

Proof. Suppose that x belongs to the left side of (17); that is, $x \in X$ and $x \notin \cup_{\alpha \in A} E_\alpha$. By definition, $x \in X$ and $x \notin E_\alpha$ for all $\alpha \in A$. Hence, $x \in E_\alpha^c$ for all $\alpha \in A$; that is, x belongs to the right side of (17). These steps are reversible. This verifies (17). A similar argument verifies

(18).

1.37 Theorem. Let X and Y be sets and $f: X \to Y$.

1. If $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets X, then
$$f\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f(E_\alpha) \quad \text{and} \quad f\left(\bigcap_{\alpha\in A} E_\alpha\right) \subseteq \bigcap_{\alpha\in A} f(E_\alpha).$$
2. If B and C are subsets of X, then $f(C \setminus \mathrm{B}) \supseteq f(C) \setminus f(B)$
3. If $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets of Y, then
$$f^{-1}\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(E_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} f^{-1}(E_\alpha).$$
4. If B and C are subsets of Y, then $f^{-1}(C \setminus B) = f^{-1}(C) \setminus f^{-1}(B)$.
5. If $E \subseteq f(X)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $f^{-1}(f(E)) \supseteq E$.

Proof.

1. By definition, $y \in f(\cup_{\alpha\in A} E_\alpha)$ if and only if $y = f(x)$ for some $x \in E_\alpha$ and $\alpha \in A$. This is equivalent to $y \in \cup_{\alpha\in A} f(E_\alpha)$. Similarly, $y \in f(\cap_{\alpha\in A} E_\alpha)$ if and only if $y = f(x)$ for some $x \in \cap_{\alpha\in A} E_\alpha$. This implies that for all $\alpha \in A$ there is an $x_\alpha \in E_\alpha$ such that $y = f(x_\alpha)$. Therefore, $y \in \cap_{\alpha\in A} f(E_\alpha)$.
2. If $y \in f(C) \setminus f(B)$, then $y = f(c)$ for some $c \in C$ but $y \neq f(b)$ for any $b \in B$. It follows that $y \in f(C \setminus B)$. Simialr arguments prove parts 3., 4., and 5.,

It is important to recognize that the set inequalities in parts i), ii), and v) can be strict unless f is 1-1 (see Exercise 1.5.6 and 1.5.7). For example, if $f(x) = x^2$, $E_1 = \{1\}$, and $E_2 = \{-1\}$, then $f(E_1 \cap E_2) = \emptyset$ is a proper subset of $f(E_1) \cap f(E_2) = \{1\}$.

1.6 Countable and Uncountable Sets

2018年10月11日 13:55

1.38 Definition

Let E be a set.

1. E is said to be *finite* if and only if either $E = \emptyset$ or there exists a 1-1 function which takes $\{1, 2, \dots, n\}$ onto E, for some $n \in \mathbb{N}$.
2. E is said to be *countable* if and only if there exists a 1-1 function which takes $\mathbb{N}$ onto E.
3. E is said to be *at most countable* if and only if E is either finite or countable.
4. E is said to be *uncountable* if and only if E is neither finite nor countable.

1.39 Remark [Cantor's Diagonalization Argument].

The open interval $(0, 1)$ is uncountable.

Strategy: Suppose to the contrary that $(0, 1)$ is countable. Then by definition, there is a function f on $\mathbb{N}$ such that $f(1), f(2), \dots$ exhausts the elements of $(0, 1)$. We could reach a contradiction if we could find a new number $x \in (0, 1)$ that is different from all the $f(k)$'s. How can we determine whether two numbers are different? One easy way is to look at their decimal expansions. For example, $0.1234 \neq 0.1254$ because they have different decimal expansions. Thus, we could find an x that has no preimage under f by making the decimal expansion of x different by at least one digit from the decimal expansion of EVERY $f(k)$.

There is a flaw in this approach that we must fix. Decimal expansions are unique except for finite decimals, which always have an alternative expansion that terminates in 9s (e.g., $0.5 = 0.49999\dots$ and $0.24 = 0.2399999\dots$) (see Exercise 2.2.10). Hence, when specifying the decimal expansion of x, we must avoid decimals that terminate in 9s.

Proof. Suppose that there is a 1-1 function f that takes $\mathbb{N}$ onto the interval $(0, 1)$. Write the numbers $f(j), j \in \mathbb{N}$, in decimal notation, using the finite expansion when possible, that is,

$$f(1) = 0.\alpha_{11}\alpha_{12}\dots,$$
$$f(2) = 0.\alpha_{21}\alpha_{22}\dots,$$
$$f(3) = 0.\alpha_{31}\alpha_{32}\dots,$$
$$\dots,$$

Where α_{ij} represents the j^{th} digit in the decimal expansion of $f(i)$ and none of these expansions terminates in 9s. Let x be the number whose decimal expansion is given by $0.\beta_1\beta_2\dots$, where

$$\beta_k := \begin{cases} \alpha_{kk} + 1 & \text{if } \alpha_{kk} \leq 5 \\ \alpha_{kk} - 1 & \text{if } \alpha_{kk} > 5 \end{cases}$$

Clearly, x is a number in $(0, 1)$ whose decimal expansion does not contain one 9, much less terminate in 9s. Since f is onto, there is a $j \in \mathbb{N}$ such that $f(j) = x$. Since we have avoided 9s, the decimal expansions of $f(j)$ and x must be identical (e.g., $a_{jj} = \beta_j := \alpha_{jj} \pm 1$). It follows that $0 = \pm 1$, a contradiction.

It is natural to ask about the countability of the sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. To answer these questions, we prove several preliminary results. First, to show that a set E is at most countable, we do not need to construct a ONE-TO-ONE function which takes $\mathbb{N}$ onto E.

1.40 Lemma.

A nonempty set E is at most countable if and only if there is a function g from $\mathbb{N}$ onto E.

Proof.

If E is countable, then by Definition 1.38ii there is a (1-1) function f from $\mathbb{N}$ onto E, so $g := f$ takes $\mathbb{N}$ onto E. If E is finite, then there is an $n \in \mathbb{N}$ and 1-1 function f that takes $\{1, 2, \dots, n\}$ onto E. Hence

$$g(j) := \begin{cases} f(j) & j \leq n \\ f(1) & j > n \end{cases}$$

Takes $\mathbb{N}$ onto E.

Conversely, suppose that g takes $\mathbb{N}$ onto E. We need to construct a function f that is 1-1 from some subset of $\mathbb{N}$ onto E. We will do this by eliminating the duplication in g. To this end, let $k_1 = 1$. If the set $E_1 := \{k \in \mathbb{N}: g(k) \neq g(k_1)\}$ is empty, then $E = \{g(k_1)\}$, thus evidently at most countable. Otherwise, let k_2 be the least element in E_1 and notice that $k_2 > k_1$.
Set $E_2 := \{k \in \mathbb{N}: g(k) \in E \setminus \{g(k_1), g(k_2)\}\}$. If E_2 is empty, then $E = \{g(k_1), g(k_2)\}$ is finite, hence at most countable. Otherwise, let k_3 be the least element in E_2. Since $g(k_3) \in E \setminus \{g(k_1), g(k_2)\}$, we have $g(k_3) \neq g(k_2)$ and $g(k_3) \neq g(k_1)$. Since g is a function, the first condition implies $k_3 \neq k_2$. Since k_2 is least in E_1, the second condition implies $k_2 < k_3$. Hence, $k_1 < k_2 < k_3$.
Continue this process. If it ever terminates, then some

$$E_j := \left\{k \in \mathbb{N}: g(k) \in E \setminus \left\{g(k_1), \dots, g\left(k_j\right)\right\}\right\}$$

Is empty, so E is finite, hence at most countable. If this process never terminates, then we generate integers $k_1 < k_2 < \cdots$ such that k_{j+1} is the least element of E_j for $j = 1, 2, \dots$.
Set $f(j) = g\left(k_j\right), j \in \mathbb{N}$. To show that f is 1-1, notice that $j \neq \ell$ implies that $k_j \neq k_\ell$, say $k_j < k_\ell$. Then $k_j \leq k_{\ell-1}$, so by construction

$$g(k_\ell) \in E \setminus \left\{g(k_1), \dots, g\left(k_j\right), \dots, g(k_{\ell-1})\right\} \subseteq E \setminus \left\{g(k_1), \dots, g\left(k_j\right)\right\}.$$

In particular, $g(k_\ell) \neq g\left(k_j\right)$; that is, $f(\ell) \neq f(j)$.

To show that f is onto, let $x \in E$. Since g is onto, choose $\ell \in \mathbb{N}$ such that $g(\ell) = x$. Since by construction $j \leq k_j$, use the Archimedean Principle to choose a $j \in \mathbb{N}$ such that $k_j > \ell$. Since k_j is the least element in E_{j-1}, it follows that $g(\ell)$ cannot belong to $E \setminus \left\{g(k_1), \dots, g\left(k_{j-1}\right)\right\}$; that is, $g(\ell) = g(k_n)$ for some $n \in [1, j-1]$. In particular, $f(n) = g(k_n) = x$.

Next, we show how set containment affects countability and use it to answer the question about countability of $\mathbb{R}$.

1.41 Theorem.

Suppose that A and B are sets.

1. If $A \subseteq B$ and B is at most countable, then A is at most countable.
2. If $A \subseteq B$ and A is uncountable, then B is uncountable.
3. $\mathbb{R}$ is uncountable.

Proof.

1. Since B is at most countable, choose by Lemma 1.40 a function g which takes $\mathbb{N}$ onto B. We may suppose that A is nonempty, hence fix an $a_0 \in A$. Then
$$f(n) := \begin{cases} g(n) & g(n) \in A \\ a_0 & g(n) \notin A \end{cases}$$
Takes $\mathbb{N}$ onto A. Hence by Lemma 1.40, A is at most countable.
2. If B were at most countable, then by part 1., A would also be at most countable, a contradiction.
3. By Remark 1.39, the interval $(0, 1)$ is an uncountable subset of $\mathbb{R}$. Thus, by part 2, $\mathbb{R}$ is uncountable.

1.42 Theorem.

Let $A_1, A_2, \dots$ be at most countable sets.

1. Then $A_1 \times A_2$ is at most countable.
2. If
$$E = \bigcup_{j=1}^{\infty} A_j := \bigcup_{j \in \mathbb{N}} A_j := \left\{x : x \in A_j \quad for\ some\ j \in \mathbb{N}\right\},$$
Then E is at most countable.

Proof.

1. By Lemma 1.40, there exist functions ϕ (respectively, ψ) which take $\mathbb{N}$ onto A_1 (respectively, onto A_2). Hence $f(n, m) := (\phi(n), \psi(m))$ takes $\mathbb{N} \times \mathbb{N}$ onto $A_1 \times A_2$. If we can construct a function g which takes $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$, then by Exercise 1.6.5a, $f \circ g$ takes $\mathbb{N}$ onto $A_1 \times A_2$. Hence by Lemma 1.40, $A_1 \times A_2$ is at most countable.

To construct the function g, plot the points of $\mathbb{N} \times \mathbb{N}$ in the plane. Notice that we can connect these lattice points with a series of parallel backward-slanted lines; for example, the first line passes through $(1,1)$, the second line passes through $(1,2)$ and $(2,1)$, and the third line passes through $(1,3)$, $(2,2)$, and $(3,1)$. This suggests a method for constructing g. Set $g(1) = (1,1)$, $g(2) = (1,2)$, $g(3) = (2,1)$, $g(4) = (3,1)$, ...

If you wish to see an explicit formula for g, observe that the nth line passes through the set of lattice points

$$(1,n), (2,n-1), (3,n-2), \dots, (n-1,2), (n,1);$$

That is, through the set of lattice points (k,j) which satisfy $k + j = n + 1$. Since the sum of integers $1 + 2 + \cdots + (n-1)$ is given by $\frac{(n-1)n}{2}$ (see Exercise 1.4.4a), there are $\frac{(n-1)n}{2}$ elements in the first $n-1$ slanted lines. Hence a function which takes $\mathbb{N}$ onto the nth slanted line is given by

$$g(j) = (\ell, n + 1 - \ell),$$

Where $j = \ell + \frac{(n-1)n}{2}$. This function is defined on all of $\mathbb{N}$ because given $j \in \mathbb{N}$, we can use the Archimedean Principle and the Well-Ordering Principle to choose n least such that $j \le \frac{n(n+1)}{2}$; that is, such that $j = \ell + \frac{(n-1)n}{2}$ for some $\ell \in [1,n]$. Thus g takes $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$.

2. By Lemma 1.40, choose functions f_j that take $\mathbb{N}$ onto $A_j, j \in \mathbb{N}$. Clearly, the function $h(k,j) \coloneqq f_k(j)$ takes $\mathbb{N} \times \mathbb{N}$ onto E. Hence the function $h \circ g$, where g is defined by (19), takes $\mathbb{N}$ onto E. We conclude by Lemma 1.40 that E is at most countable.

1.43 Remark.

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but the set of irrationals is uncountable.

Proof.

$\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$ and $\mathbb{Q} = \cup_{n=1}^{\infty} \left\{\frac{p}{n} : p \in \mathbb{Z}\right\}$ are both countable by Theorem 1.42ii.

If $\mathbb{R} \setminus \mathbb{Q}$ were countable, then $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$ would also be countable, a contradiction of Theorem 1.41iii.

Chapter 2 Sequences in ℝ

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2.1 Limits of Sequences

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2.1 Definition

A sequence of real numbers $\{x_n\}$ is said to converge to a real number $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ϵ) such that

$n \geq N \quad$ implies $\left|x_n - a\right| < \epsilon$

2.2 Example

1. Prove that $\frac{1}{n} \to 0$ as $n \to \infty$.
2. If $x_n \to 2$, prove that $\frac{2x_n+1}{x_n} \to \frac{5}{2}$ as $n \to \infty$

Proof.

1. Let $\epsilon > 0$. Use the Archimedean Principle to choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. By taking the reciprocal of this inequality, we see that $n \geq N$ implies $\frac{1}{n} \leq \frac{1}{N} < \epsilon$. Since $\frac{1}{n}$ are all positive, it follows that $\left|\frac{1}{n}\right| < \epsilon$ for all $n \geq N$.

   Strategy for 2:

   By definition, we must show that
   $$\frac{2x_n + 1}{x_n} - \frac{5}{2} = \frac{2 - x_n}{2x_n}$$
   Is small for large n. The numerator of this last fraction will be small for large n since $x_n \to 2$, as $n \to \infty$. What about the denominator? Since $x_n \to 2$, x_n will be greater than 1 for large n, so $2x_n$ will be greater than 2 for large n. Since we made n large twice, we will make two restrictions to determine the N that corresponds to ϵ in Definition 2.1. Let's try to write all this down carefully to be sure that it works out.
2. Let $\epsilon > 0$. Since $x_n \to 2$, apply Definition 2.1 to this $\epsilon > 0$ to choose $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $\left|x_n - 2\right| < \epsilon$. Next, apply Definition 2.1 with $\epsilon = 1$ to choose N_2 such that $n \geq N_2$ implies $\left|x_n - 2\right| < 1$. By the Fundamental Theorem of Absolute Values, we have $n \geq N_2$ implies $x_n > 1$ (i.e., $2x_n > 2$).

   Set $N = \max\{N_1, N_2\}$ and suppose that $n \geq N$. Since $n \geq N_1$, we have $\left|2 - x_n\right| = \left|x_n - 2\right| < \epsilon$. Since $n \geq N_2$, we have $0 < \frac{1}{2x_n} < \frac{1}{2} < 1$. It follows that
   $$\left|\frac{2x_n + 1}{x_n} - \frac{5}{2}\right| = \frac{\left(\left|2 - x_n\right|\right)}{2x_n} < \frac{\epsilon}{2x_n} < \epsilon$$
   For all $n \geq N$.

2.3 Example.

The sequence $\{(-1)^n\}_{n \in \mathbb{N}}$ has no limit.

Proof. Suppose that $(-1)^n \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$. Given $\epsilon = 1$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|(-1)^n - a| < \epsilon$**.** For n odd this implies $|1 + a| = |-1 - a| < 1$, and for n even this implies $|1 - a| < 1$. Hence

$$2 = |1 + 1| \leq |1 - a| + |1 + a| < 1 + 1 = 2;$$

That is, $2 < 2$, a contradiction.

2.4 Remark.

A sequence can have at most one limit.

Proof.

Suppose that $\{x_n\}$ converges to both a and b. By definition, given $\epsilon > 0$, there is an integer N such that $n \geq N$ implies $\left|x_n - a\right| < \frac{\epsilon}{2}$ and $\left|x_n - b\right| < \frac{\epsilon}{2}$. Thus it follows from the triangle inequality that

$$|a - b| \leq \left|a - x_n\right| + \left|x_n - b\right| < \epsilon$$

That is, $|a - b| < \epsilon$ for all $\epsilon > 0$. We conclude, by Theorem 1.9, that $a = b$.

2.5 Definition

By a *subsequence* of a sequence $\{x_n\}_{n \in \mathbb{N}}$, we shall mean a sequence of the form $\left\{x_{n_k}\right\}_{k \in \mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$.

2.6 Remark

If $\{x_n\}_{n\in\mathbb{N}}$ converges to a and $\{x_{n_k}\}_{k\in\mathbb{N}}$ is any subsequence of $\{x_n\}_{n\in\mathbb{N}}$, then x_{n_k} converges to a as $k \to \infty$.

Proof.

Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - a| < \epsilon$. Since $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$, it is easy to see by induction that $n_k \geq k$ for all $k \in \mathbb{N}$. Hence, $k \geq N$ implies $|x_{n_k} - a| < \epsilon$; that is, $x_{n_k} \to a$ as $k \to \infty$.

2.7 Definition

Let $\{x_n\}$ be a sequence of real numbers.

1. The sequence $\{x_n\}$ is said to be bounded above if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded above.
2. The sequence $\{x_n\}$ is said to be bounded below if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded below.
3. $\{x_n\}$ is said to be bounded if and only if it is bounded both above and below.

Combining Definitions 2.7 and 1.10, we see that $\{x_n\}$ is bounded above (respectively, below) if and only if there is an $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$ (respectively, if and only if there is an $m \in \mathbb{R}$ such that $x_n \geq m$ for all $n \in \mathbb{R}$). It is easy to check (see Exercise 2.1.4) that $\{x_n\}$ is bounded if and only if there is a $C > 0$ such that $|x_n| \leq C$ for all $n \in \mathbb{N}$. In this case we shal say that $\{x_n\}$ is bounded, or dominated, by C.

2.8 Theorem.

Every convergent sequence is bounded.

Strategy: The idea behind the proof is simple (see Figure 2.1). Suppose that $x_n \to a$ as $n \to \infty$. By definition, for large N the sequence $x_N, x_{N+1}, \ldots$ must be close to a, hence bounded. Since the finite sequence $x_1, \ldots, x_{N-1}$ is also bounded, it should follow that the whole sequence is bounded. We now make this precise.

Proof.

Given $\epsilon = 1$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - a| < 1$. Hence by the triangle inequality, $|x_n| < 1 + |a|$ for all $n \geq N$. On the other hand, if $1 \leq n \leq N$, then

$$|x_n| \leq M := \max\{|x_1|, |x_2|, \ldots, |x_N|\}.$$

Therefore, $\{x_n\}$ is dominated by $\max\{M, 1 + |a|\}$.

2.2 Limit Theorems

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2.9 Theorem [Squeeze Theorem].
Suppose that $\{x_n\}, \{y_n\}$ and $\{w_n\}$ are real sequences.

1. If $x_n \to a$ and $y_n \to a$ (the SAME a) as $n \to \infty$, and if there is an $N_0 \in \mathbb{N}$ such that
   $$x_n \le w_n \le y_n \quad for\ n \ge N_0,$$
   Then $w_n \to a$ as $n \to \infty$.
2. If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

Proof.

1. Let $\epsilon > 0$. Since x_n and y_n converge to a, use Definition 2.1 and Theorem 1.6 to choose $N_1, N_2 \in \mathbb{N}$ such that $n \ge N_1$ implies $-\epsilon < x_n - a < \epsilon$ and $n \ge N_2$ implies $-\epsilon < y_n - a < \epsilon$. Set $N = \max\{N_0, N_1, N_2\}$. If $n \ge N$, we have by hypothesis and the choise of N_1 and N_2 that
   $$a - \epsilon < x_n \le w_n \le y_n < a + \epsilon;$$
   That is, $|w_n - a| < \epsilon$ for $n \ge N$. We conclude that $w_n \to a$ as $n \to \infty$.
2. Suppose that $x_n \to 0$ and that there is an $M > 0$ such that $|y_n| \le M$ for $n \in \mathbb{N}$. Let $\epsilon > 0$ and choose an $N \in \mathbb{N}$ such that $n \ge N$ implies $|x_n| < \frac{\epsilon}{M}$. Then $n \ge N$ implies
   $$|x_n y_n| < M \frac{\epsilon}{M} = \epsilon$$
   We conclude that $x_n y_n \to 0$ as $n \to \infty$.

2.10 Example.
Find $\lim_{n\to\infty} 2^{-n} \cos(n^3 - n^2 + n - 13)$.
Solution.
The factor $\cos(n^3 - n^2 + n - 13)$ looks intimidating, but it is superfluous for finding the limit of this sequence. Indeed, since $|\cos x| \le 1$ for all $x \in \mathbb{R}$, the sequence $\{2^{-n} \cos(n^3 - n^2 + n - 13)\}$ is dominated by 2^{-n}. Since $2^n > n$, it is clear by Example 2.2i and the Squeeze Theorem that both $2^{-n} \to 0$ and $2^{-n} \cos(n^3 - n^2 + n - 13) \to 0$ as $n \to \infty$.

2.11 Theorem.
Let $E \subset \mathbb{R}$. If E has a finite supremum (respectively, a finite infimum), then there is a sequence $x_n \in E$ such that $x_n \to \sup E$ (respectively, a sequence $y_n \in E$ such that $y_n \to \inf E$) as $n \to \infty$.
Proof.
Suppose that E has a finite supremum. For each $n \in \mathbb{N}$, choose (by the Approximation Property for Suprema) an $x_n \in E$ such that $\sup E - \frac{1}{n} < x_n \le \sup E$. Then by the Squeeze Theorem and Example 2.2i, $x_n \to \sup E$ as $n \to \infty$. Similarly, there is a sequence $y_n \in E$ such that $y_n \to \inf E$.

2.12 Theorem.
Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are convergent, then

1. $\lim\limits_{n\to\infty}(x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$
2. $\lim\limits_{n\to\infty}(\alpha x_n) = \alpha \lim\limits_{n\to\infty} x_n$,
   And
3. $\lim\limits_{n\to\infty}(x_n y_n) = \left(\lim\limits_{n\to\infty} x_n\right)\left(\lim\limits_{n\to\infty} y_n\right)$.
   If, in addition, $y_n \ne 0$ and $\lim_{n\to\infty} y_n \ne 0$, then
4. $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{\left(\lim\limits_{n\to\infty} x_n\right)}{\left(\lim\limits_{n\to\infty} y_n\right)}$.

(In particular, all these limits exist.)
Proof.
Suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

1. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. Thus $n \geq N$ implies
$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
2. It suffices to show that $\alpha x_n - \alpha x \to 0$ as $n \to \infty$. But $x_n - x \to 0$ as $n \to \infty$, hence by the Squeeze Theorem, $\alpha(x_n - x) \to 0$ as $n \to \infty$.
3. By Theorem 2.8, the sequence $\{x_n\}$ is bounded. Hence by the Squeeze Theorem the sequence $\{x_n(y_n - y)\}$ and $\{(x_n - x)y\}$ both converge to 0. Since
$$x_n y_n - xy = x_n(y_n - y) + (x_n - x)y,$$
It follows from part 1) that $x_n y_n \to xy$ as $n \to \infty$. A similar argument establishes part 4) (see Exercise 2.2.4)

2.13 Example.
Find $\lim_{n\to\infty} \frac{n^3+n^2-1}{1-3n^2}$.
Solution.
Multiplying the numerator and denominator by $\frac{1}{n^3}$, we find that
$$\frac{n^3 + n^2 - 1}{1 - 3n^3} = \frac{\left(1 + \left(\frac{1}{n}\right) - \left(\frac{1}{n^3}\right)\right)}{\frac{1}{n^3} - 3}$$
By Example 2.2i and Theorem 2.12iii, $\frac{1}{n^k} = \left(\frac{1}{n}\right)^k \to 0$, as $n \to \infty$, for any $k \in \mathbb{N}$. Thus by Theorem 2.12i, ii, and iv,
$$\lim_{n\to\infty} \frac{n^3 + n^2 - 1}{1 - 3n^3} = \frac{1 + 0 - 0}{0 - 3} = -\frac{1}{3}$$

2.14 Definition.
Let $\{x_n\}$ be a sequence of real numbers.
1. $\{x_n\}$ is said to *diverge* to $+\infty$ (notation: $x_n \to +\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = +\infty$) if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
$$n \geq N \text{ implies } x_n > M.$$
2. $\{x_n\}$ is said to *diverge* to $-\infty$ (notation: $x_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$) if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that
$$n \geq N \text{ implies } x_n < M.$$

2.15 Theorem.
Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n \to +\infty$ (respectively, $x_n \to -\infty$) as $n \to \infty$.
1. If y_n is bounded below (respectively, y_n is bounded above), then
$$\lim_{n\to\infty}(x_n + y_n) = +\infty \qquad \left(\textit{respectively}, \lim_{n\to\infty}(x_n + y_n) = -\infty\right).$$
2. If $\alpha > 0$, then
$$\lim_{n\to\infty}(\alpha x_n) = +\infty \qquad \left(\textit{respectively}, \lim_{n\to\infty}(\alpha x_n) = -\infty\right).$$
3. If $y_n > M_0$ for some $M_0 > 0$ and all $n \in \mathbb{N}$, then
$$\lim_{n\to\infty}(x_n y_n) = +\infty \qquad \left(\textit{respectively}, \lim_{n\to\infty}(x_n y_n) = -\infty\right).$$
4. If $\{y_n\}$ is bounded and $x_n \neq 0$, then
$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

Proof.
We suppose for simplicity that $x_n \to +\infty$ as $n \to \infty$.
1. By hypothesis, $y_n \geq M_0$ for some $M_0 \in \mathbb{R}$. Let $M \in \mathbb{R}$ and set $M_1 = M - M_0$. Since $x_n \to +\infty$, choose $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n > M_1$. Then $n \geq N$ implies $x_n + y_n > M_1 + M_0 = M$.
2. Let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{\alpha}$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n > M_1$. Since $\alpha > 0$, we conclude that $\alpha x_n > \alpha M_1 = M$ for all $n \geq N$.

3. Let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{M_0}$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n y_n > M_1 M_0 = M$.
4. Let $\epsilon > 0$. Choose $M_0 > 0$ such that $|y_n| \leq M_0$ and $M_1 > 0$ so large that $\frac{M_0}{M_1} < \epsilon$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n > M_1$. Then $n \geq N$ implies $x_n > M_1$. Then $n \geq N$ implies
$$\left|\frac{y_n}{x_n}\right| = \frac{|y_n|}{x_n} < \frac{M_0}{M_1} < \epsilon.$$

If we adopt the conventions

$$x + \infty = \infty, \quad x - \infty = -\infty, \quad x \in \mathbb{R}$$
$$x \cdot \infty = \infty, \quad x \cdot (-\infty) = -\infty, \quad x > 0,$$
$$x \cdot \infty = -\infty, \quad x \cdot (-\infty) = \infty, \quad x < 0,$$
$$\infty + \infty = \infty, \quad -\infty - \infty = -\infty,$$
$$\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty, \quad and \ \infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty,$$

2.16 Corollary.

Let $\{x_n\}, \{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$, as $n \to \infty$, then
$$\lim_{n\to\infty} (x_n + y_n) = x + y$$
Provided that the right side is not of the form $\infty - \infty$, and
$$\lim_{n\to\infty} (\alpha x_n) = \alpha x, \lim_{n\to\infty} (x_n y_n) = xy$$
Provided that none of these products is of the form $0 \cdot \pm\infty$.

We have avoided the cases $\infty - \infty$ and $0 \cdot \pm\infty$ because they are "indeterminate". For a discussion of indeterminates forms, see l'Hospital's Rule in Section 4.4

2.17 Theorem. [Comparison Theorem].

Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that
$$x_n \leq y_n \ \ for \ n \geq N_0,$$
Then
$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n .$$
In particular, if $x_n \in [a, b]$ converges to some point c, then c must belong to $[a, b]$.

Proof. Suppose that the first statement is false; that is (1) holds but $x \coloneqq \lim_{n\to\infty} x_n$ is greater than $y \coloneqq \lim_{n\to\infty} y_n$. Set $\epsilon = \frac{x-y}{2}$. Choose $N_1 > N_0$ such that $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon$ for $n \geq N_1$. Then for such an n,
$$x_n > x - \epsilon = x - \left(\frac{x-y}{2}\right) = y + \left(\frac{x-y}{2}\right) = y + \epsilon > y_n,$$
Which contradicts (1). This proves the first statement.

We conclude by noting that the second statement follows from the first, since $a \leq x_n \leq b$ implies $a \leq c \leq b$.

One way to remember this result is that it says the limit of an inequality is the inequality of the limits, provided these limits exist. We shall call this process "taking the limit of an inequality". Since $x_n < y_n$ implies $x_n \leq y_n$, the Comparison Theorem contains the following corollary: If $\{x_n\}$ and $\{y_n\}$ are convergent real sequences, then
$$x_n < y_n, \quad n \geq N_0, \quad \text{imply} \quad \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n .$$
It is important to notice that this result is false if $\leq$ is replaced by $<$; that is
$$x_n < y_n, \quad n \geq N_0, \quad \text{does NOT imply that} \ \lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n .$$
For example, $\frac{1}{n^2} < \frac{1}{n}$, but the limits of these sequences are equal.

2.3 Bolzano-Weierstrass Theorem

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2.18 Definition.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.

1. $\{x_n\}$ is said to be increasing (respectively, strictly increasing) if and only if $x_1 \le x_2 \le \cdots$ (respectively, $x_1 < x_2 < \cdots$)
2. $\{x_n\}$ is said to be decreasing (respectively, strictly decreasing) if and only if $x_1 \ge x_2 \ge \cdots$ (respectively, $x_1 > x_2 > \cdots$).
3. $\{x_n\}$ is said to be monotone if and only if it is either increasing or decreasing.

(Some authors call decreasing sequences nonincreasing and increasing sequences nondecreasing.)

2.19 Theorem. [Monotone Convergence Theorem].

If $\{x_n\}$ is increasing and bounded above, or if $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

Proof.

Suppose first that $\{x_n\}$ is increasing and bounded above. By the Completeness Axiom, the supremum $a \coloneqq \sup\{x_n : n \in \mathbb{N}\}$ exists and is finite. Let $\epsilon > 0$. By the Approximation Property for Suprema, choose $N \in \mathbb{N}$ such that

$$a - \epsilon < x_N \le a.$$

Since $x_N \le x_n$ for $n \ge N$ and $x_n \le a$ for all $n \in \mathbb{N}$, it follows that $a - \epsilon < x_n \le a$ for all $n \ge N$. In particular, $x_n \uparrow a$ as $n \to \infty$.

If $\{x_n\}$ is decreasing with infimum $b \coloneqq \inf\{x_n : n \in \mathbb{N}\}$, then $\{-x_n\}$ is increasing with supremum $-b$ (see Theorem 1.20). Hence, by the first case and Theorem 2.12ii,

$$b = -(-b) = -\lim_{n\to\infty}(-x_n) = \lim_{n\to\infty} x_n\,.$$

The Monotone Convergence Theorem is used most often to show that a limit exists. Once existence has been established, it is often easy to find the value of that limit by using Theorems 2.9 and 2.12. The following examples illustrate this fact.

2.20 Example.

If $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.

Proof.

It suffices to prove that $|a|^n \to 0$ as $n \to \infty$. First, we notice that $|a|^n$ is monotone decreasing since by the Multiplicative Property, $|a| < 1$ implies $|a|^{n+1} < |a|^n$ for all $n \in \mathbb{N}$. Next, we observe that $|a|^n$ is bounded below (by 0). Hence by the Monotone Convergence Theorem, $L \coloneqq \lim_{n\to\infty}|a|^n$ exists.

Take the limit of the algebraic identity $|a|^{n+1} = |a| \cdot |a|^n$, as $n \to \infty$. By Remark 2.6 and Theorem 2.12, we obtain $L = |a| \cdot L$. Thus either $L = 0$ or $|a| = 1$. Since $|a| < 1$ by hypothesis, we conclude that $L = 0$.

2.21 Example.

If $a > 0$, then $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Proof.

We consider three cases.

Case 1. $a = 1$. Then $a^{\frac{1}{n}} = 1$ for all $n \in \mathbb{N}$, and it follows that $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Case 2. $a > 1$. We shall apply the monotone Convergence Theorem. To show that $\left\{a^{\frac{1}{n}}\right\}$ is decreasing, fix $n \in \mathbb{N}$ and notice that $a > 1$ implies $a^{n+1} > a^n$. Taking the $n(n+1)$st root of this inequality, we obtain $a^{\frac{1}{n}} > a^{\frac{1}{n+1}}$; that is, $a^{\frac{1}{n}}$ is decreasing. Since $a > 1$ implies $a^{\frac{1}{n}} > 1$, it follows that $a^{\frac{1}{n}}$ is decreasing and bounded below. Hence, by the Monotone Convergence Theorem, $L \coloneqq \lim_{n\to\infty} a^{\frac{1}{n}}$ exists. To find its value, take the limit of the identity $\left(a^{\frac{1}{2n}}\right)^2 = a^{\frac{1}{n}}$ as $n \to \infty$. We

obtain $L^2 = L$; that is, $L = 0$ or 1. Since $a^{\frac{1}{n}} > 1$, the Comparison Theorem shows that $L \geq 1$. Hence $L = 1$.
Case 3. $0 < a < 1$. Then $\frac{1}{a} > 1$. It follows from Theorem 2.21 and Case 2 that

$$\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{a^{\frac{1}{n}}}} = \frac{1}{\lim_{n\to\infty}\left(\frac{1}{a}\right)^{\frac{1}{n}}} = 1.$$

Next, we introduce a monotone property for sequences of sets.

2.22 Definition.
A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be nested if and only if

$$I_1 \supseteq I_2 \supseteq \cdots.$$

2.23 Theorem. [Nested Interval Property].
If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then $E \coloneqq \bigcap_{n=1}^{\infty} I_n$ is nonempty. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then E is a single point.

Proof.
Let $I_n = [a_n, b_n]$. Since $\{I_n\}$ is nested, the real sequence $\{a_n\}$ is increasing and bounded above by b_1, and $\{b_n\}$ is decreasing and bounded below by a_1 (see Figure 2.2). Thus by Theorem 2.19, there exist $a, b \in \mathbb{R}$ such that $a_n \uparrow a$ and $b_n \downarrow b$ as $n \to \infty$. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, it also follows from the Comparison Theorem that $a_n \leq a \leq b \leq b_n$. Hence, a number x belongs to I_n for all $n \in \mathbb{N}$ if and only if $a \leq x \leq b$; that is, if and only if $x \in [a, b]$. In particular, any $x \in [a, b]$ belongs to all the I_n's.

We have proved that there is exactly one number that belongs to all the I_n's if and only if $a = b$. But if $|I_n| \to 0$ as $n \to \infty$, then $b_n - a_n \to 0$ as $n \to \infty$. Hence, by Theorem 2.12, a does equal b when $|I_n| \to 0$ as $n \to \infty$.

2.24 Remark.
The Nested Interval Property might not hold if "closed" is omitted.

Proof.
The intervals $I_n = \left(0, \frac{1}{n}\right)$, $n \in \mathbb{N}$, are bounded and nested but not closed. If there were an $x \in I_n$ for all $n \in \mathbb{N}$, then $0 < x < \frac{1}{n}$; that is, $n < \frac{1}{x}$ for all $n \in \mathbb{N}$. Since this contradicts the Archimedean Principle, it follows that the intervals I_n have no point in common.

2.25 Remark.
The Nested Interval Property might not hold if "bounded" is omitted.

Proof.
The intervals $I_n = [n, \infty)$, $n \in \mathbb{N}$ are closed and nested but not bounded. Again, they have no point in common.

We are now prepared to establish the main result of this section.

2.26 Theorem. [Bolzano-Weierstrass Theorem].
Every bounded sequence of real numbers has a convergent subsequence.

Proof.
We begin with a general observation. Let $\{x_n\}$ be any sequence. If $E = A \cup B$ are sets and E contains x_n for infinitely many values of n, then at least one of the sets A or B also contains x_n for infinitely many values of n. (If not, then E contains x_n for only finitely many n, a contradiction.)
Let $\{x_n\}$ be a bounded sequence. Choose $a, b \in \mathbb{R}$ such that $x_n \in [a, b]$ for all $n \in \mathbb{N}$, and set $I_0 = [a, b]$. Divide I_0 into two halves, say $I' = \left[a, \frac{a+b}{2}\right]$ and $I'' = \left[\frac{a+b}{2}, b\right]$. Since $I_0 = I' \cup I''$, at least

one of these half-intervals contains x_n for infinitely many n. Call it I_1, and choose $n_1 > 1$ such that $x_{n_1} \in I_1$. Notice that $|I_1| = \frac{|I_0|}{2} = \frac{b-a}{2}$.
Suppose that closed intervals $I_0 \supset I_1 \supset \cdots \supset I_m$ and natural numbers $n_1 < n_2 < \cdots < n_m$ have been chosen such that for each $0 \leq k \leq m$,

$$|I_k| = \frac{b-a}{2^k}, \qquad x_{n_k} \in I_k, \qquad \text{and } x_n \in I_k \text{ for infinitely many } n$$

To choose I_{m+1}, divide $I_m = [a_m, b_m]$ into two halves, say $I' = \left[a_m, \frac{a_m+b_m}{2}\right]$ and $I'' = \left[\frac{a_m+b_m}{2}, b_m\right]$. Since $I_m = I' \cup I''$, at leasst one of these half-intervals contains x_n for infinitely many n. Call it I_{m+1}, and choose $n_{m+1} > n_m$ such that $x_{n_{m+1}} \in I_{m+1}$. Since

$$|I_{m+1}| = \frac{|I_m|}{2} = \frac{b-a}{2^{m+1}},$$

It follows by induction that there is a nested sequence $\{I_k\}_{k\in\mathbb{N}}$ of nonempty closed bounded intervals that satisfy (2) for all $k \in \mathbb{N}$.
By the Nested Interval Property, there is an $x \in \mathbb{R}$ that belongs to I_k for all $k \in \mathbb{N}$. Since $x \in I_k$, we have by (2) that

$$0 \leq |x_{n_k} - x| \leq |I_k| \leq \frac{b-a}{2^k}$$

For all $k \in \mathbb{N}$. Hence by the Squeeze Theorem, $x_{n_k} \to x$ as $k \to \infty$.

2.4 Cauchy Sequences

2018年10月28日 15:23

2.27 Definition.
A sequence of points $x_n \in \mathbb{R}$ is said to be *Cauchy* (in $\mathbb{R}$) if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n, m \geq N \quad \text{imply} \quad |x_n - x_m| < \epsilon.$$

2.28 Remark.
If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.
Proof.
Suppose that $x_n \to a$ as $n \to \infty$. Then by definition, given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - a| < \frac{\epsilon}{2}$ for all $n \geq N$. Hence if $n, m \geq N$, it follows from the triangle inequality that

$$|x_n - x_m| \leq |x_n - a| + |x_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

2.29 Theorem. [Cauchy].
Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ converges (to some point a in $\mathbb{R}$).

Strategy: By Remark 2.28, we need only show that every Cauchy sequence converges. Suppose that $\{x_n\}$ is Cauchy. Since the x_n's are near each other, the sequence $\{x_n\}$ should be bounded. Hence, by the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subsequence, say x_{n_k}. This means that for large k, the x_{n_k}'s are near some point $a \in \mathbb{R}$. But since $\{x_n\}$ is Cauchy, the x_n's should be near the x_{n_k}'s for large n, hence also near a. Thus the full sequence should converge to that same point a. Here are the details.

Proof.
Suppose that $\{x_n\}$ is Cauchy. Given $\epsilon = 1$, choose $N \in \mathbb{N}$ such that $|x_N - x_m| < 1$ for all $m \geq N$. By the triangle inequality,

$$|x_m| < 1 + |x_N| \quad \text{for } m \geq N.$$

Therefore, $\{x_n\}$ is bounded by $M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, 1 + |x_N|\}$.
By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to a$ as $k \to \infty$. Let $\epsilon > 0$. Since x_n is Cauchy, choose $N_1 \in \mathbb{N}$ such that

$$n, m \geq N_1 \quad \text{imply} \quad |x_n - x_m| < \frac{\epsilon}{2}.$$

Since $x_{n_k} \to a$ as $k \to \infty$, choose $N_2 \in \mathbb{N}$ such that

$$k \geq N_2 \quad \text{implies} \quad |x_{n_k} - a| < \frac{\epsilon}{2}.$$

Fix $k \geq N_2$ such that $n_k \geq N_1$. Then

$$|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \epsilon$$

For all $n \geq N_1$. Thus $x_n \to a$ as $n \to \infty$.

The result is extremely useful because it is often easier to show that a sequence is Cauchy than to show that it converges. The reason for this, as the following example shows, is that we can prove that a sequence is Cauchy even when we have no idea what its limit is.
2.30 Example.
Prove that any real sequence $\{x_n\}$ that satisfies

$$|x_n - x_{n+1}| \leq \frac{1}{2^n}, \qquad n \in \mathbb{N},$$

is convergent.

Proof.

If $m > n$, then

$$\begin{aligned}|x_n - x_m| &= |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m| \\ &\le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &\le \frac{1}{2^n} + \cdots + \frac{1}{2^{m-1}} \\ &= \frac{1}{2^{n-1}} \sum_{k=1}^{m-n} \frac{1}{2^k} = \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^{m-n}}\right).\end{aligned}$$

(The last step uses Exercise 1.4.4c, for $a = 2$.) It follows that $|x_n - x_m| < \frac{1}{2^{n-1}}$ for all integers $m > n \ge 1$. But given $\epsilon > 0$, we can choose $N \in \mathbb{N}$ so large that $n \ge N$ implies $\frac{1}{2^{n-1}} < \epsilon$. We have proved that $\{x_n\}$ is Cauchy. By Theorem 2.29, therefore, it converges to some real number.

The following result shows that a sequence is not necessarily Cauchy just because x_n is near x_{n+1} for large n.

2.31 Remark.
A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

Proof.
Consider the sequence $x_n := \log n$. By basic properties of logarithms (see Exercise 5.3.7),

$$x_{n+1} - x_n = \log(n+1) - \log n = \log\left(\frac{n+1}{n}\right) \to \log 1 = 0$$

As $n \to \infty$. $\{x_n\}$ cannot be Cauchy, however, because it does not converge; in fact, it diverges to $+\infty$ as $n \to \infty$.

*2.5 Limits Supremum and Infimum

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The Section is not complete. Space holder.

2.32 Definition.
Let $\{x_n\}$ be a real sequence. Then the limit supremum of $\{x_n\}$ is the extended real number

Chapter 3 Functions on $\mathbb{R}$

2018年10月30日 12:12

3.1 Two-sided Limits

2018年10月30日 12:13

3.1 Definition.
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be real function defined everywhere on I expect possibly at a. Then $f(x)$ is said to converge to L, as x approaches a, if and only if for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

In this case we write

$$L = \lim_{x \to a} f(x) \quad \text{or } f(x) \to L \ \text{ as } x \to a,$$

And call L the limit of $f(x)$ as x approaches a.

3.2 Example.
Suppose that $f(x) = mx + b$, where $m, b \in \mathbb{R}$. Prove that

$$f(a) = \lim_{x \to a} f(x)$$

For all $a \in \mathbb{R}$.
Proof.
If $m = 0$, there is nothing to prove. Otherwise, given $\epsilon > 0$, set $\delta = \frac{\epsilon}{|m|}$. If $|x - a| < \delta$, then

$$|f(x) - f(a)| = |mx + b - (ma + b)| = |m||x - a| < |m|\delta = \epsilon.$$

Thus by definition, $f(x) \to f(a)$ as $x \to a$.
Sometimes, in order to determine δ, we must break $f(x) - L$ into two factors, replacing the less important factor by an upper bound.

3.3 Example.
If $f(x) = x^2 + x - 3$, prove that $f(x) \to -1$ as $x \to 1$.
Proof.
Let $\epsilon > 0$ and set $L = -1$. Notice that

$$f(x) - L = x^2 + x - 2 = (x - 1)(x + 2).$$

If $0 < \delta \leq 1$, then $|x - 1| < \delta$ implies $0 < x < 2$, so by the triangle inequality, $|x + 2| \leq |x| + 2 < 4$. Set $\delta = \min\left\{1, \frac{\epsilon}{4}\right\}$. It follows that $|x - 1| < \delta$, then

$$|f(x) - L| = |x - 1||x + 2| < 4|x - 1| < 4\delta < \epsilon.$$

Thus by definition, $f(x) \to L$ as $x \to 1$.

Before continuing, we would like to draw your attention to two features of Definition 3.1:
Assumption 1. The interval I is open; **Assuption 2.** $0 < |x - a|$. If $I = (c, d)$ is an open interval and $\delta_0 := \min\{a - c, d - a\}$, then $|x - a| < \delta_0$ implies $x \in I$. Hence, Assuption 1 guarentees that for $\delta > 0$ sufficiently small, $f(x)$ is defined for all $x \neq a$ satisfying $|x - a| < \delta$ (i.e., on BOTH sides of a). Since $|x - a| > 0$ is equivalent to $x \neq a$, Assumption 2 guarentees that f can have a limit at a without being defined at a. (This will be cricial for defining derivatives later.)
The next result shows that even when a function f is defined at a, the value of the limit of f at a is, in general, independent of the value $f(a)$.

3.4 Remark.
Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined everywhere on I except possibly at a. If $f(x) = g(x)$ for all $x \in I \setminus \{a\}$ and $f(x) \to L$ as $x \to a$, then $g(x)$ also has a limit as $x \to a$, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x).$$

Proof.
Let $\epsilon > 0$ and choose $\delta > 0$ small enough so that (1) holds and $|x - a| < \delta$ implies $x \in I$.
Suppose that $0 < |x - a| < \delta$. We have $f(x) = g(x)$ by hypothesis and $|f(x) - L| < \epsilon$ by (1). It follows that $|g(x) - L| < \epsilon$.

Thus to prove that a function f has a limit, we may begin by simplifying f algebraically, even when that algebra is invalid at finitely many points.

3.5 Example.
Prove that

$$g(x) = \frac{x^3 + x^2 - x - 1}{x^2 - 1}$$

Has a limit as $x \to 1$.

Proof.
Set $f(x) = x + 1$ and observe by Example 3.2 that $f(x) \to 2$ as $x \to 1$. Since

$$g(x) = \frac{x^3 + x^2 - x - 1}{x^2 - 1} = \frac{(x+1)(x^2-1)}{x^2 - 1} = f(x)$$

For $x \neq \pm 1$, it follows from Remark 3.4 that $g(x)$ has a limit as $x \to 1$ (and that limit is 2).

3.6 Theorem. [Sequential Characterization of Limits].

Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined everywhere on I except possibly at a. Then

$$L = \lim_{x \to a} f(x)$$

Exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ which converges to a as $n \to \infty$.

Proof.
Suppose that f converges to L as x approaches a. Then given $\epsilon > 0$ there is a $\delta > 0$ such that (1) holds. If $x_n \in I \setminus \{a\}$ converges to a as $n \to \infty$, then choose an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - a| < \delta$. Since $x_n \neq a$, it follows from (1) that $|f(x_n) - L| < \epsilon$ for all $n \geq N$. Therefore, $f(x_n) \to L$ as $n \to \infty$.

Conversely, suppose that $f(x_n) \to L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ which converges to a. If f does not converge to L as x approaches a, then there is an $\epsilon > 0$ (call it ϵ_0) such that the implication $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon_0$ does not hold for any $\delta > 0$. Thus, for each $\delta = \frac{1}{n}, n \in \mathbb{N}$, there is a point $x_n \in I$ which satisfies two conditions: $0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon_0$. Now the first condition and the Squeeze Theorem (Theorem 2.9) imply that $x_n \neq a$ and $x_n \to a$ so by hypothesis, $f(x_n) \to L$, as $n \to \infty$. In particular, $|f(x_n) - L| < \epsilon_0$ for n large, which contradicts the second condition.

3.7 Example.
Prove that

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Has no limit as $x \to 0$.

Proof.
By examining the graph of $y = f(x)$ (see Figure 3.1), we are led to consider two extremes:

$$a_n := \frac{2}{(4n+1)\pi} \text{ and } b_n := \frac{2}{(4n+3)\pi}, \qquad n \in \mathbb{N}.$$

Clearly, both a_n and b_n converge to 0 as $n \to \infty$. On the other hand, since $f(a_n) = 1$ and $f(b_n) = -1$ for all $n \in \mathbb{N}$, $f(a_n) \to 1$ and $f(b_n) \to -1$ as $n \to \infty$. Thus by Theorem 3.6, the limit of $f(x)$, as $x \to 0$, cannot exist.

Theorem 3.6 also allows us to translate results about limits of sequences to results about limits of functions. The next three theorems illustrate this principle.
Before stating these results, we introduce an algebra of functions. Suppose that $f, g: E \to \mathbb{R}$. For each $x \in E$, the *pointwise sum*, $f + g$, of f and g is defined by

$$(f+g)(x) := f(x)+g(x),$$
The scalar product, αf, of a scalar $\alpha \in \mathbb{R}$ with f, by
$$(\alpha f)(x) := \alpha f(x),$$
The pointwise product, fg, of f and g, by
$$(fg)(x) := f(x)g(x),$$
And (when $g(x) \neq 0$) the pointwise quotient, $\frac{f}{g}$, of f and g, by
$$\left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}.$$
The following result is a function analogue of Theorem 2.12

3.8 Theorem.
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined everywhere on I except possibly at a. If $f(x)$ and $g(x)$ converge as x approaches a, then so do $(f+g)(x)$, $(fg)(x)$, $(\alpha f)(x)$, and $\left(\frac{f}{g}\right)(x)$ (when the limit of $g(x)$ is nonzero). If fact,
$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x),$$
$$\lim_{x\to a}(\alpha f)(x) = \alpha \lim_{x\to a} f(x),$$
$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x),$$
And (when the limit of $g(x)$ is nonzero)
$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$
Proof.
Let
$$L := \lim_{x\to a} f(x) \quad \text{and} \quad M := \lim_{x\to a} g(x).$$
If $x_n \in I \setminus \{a\}$ converges to a, then by Theorem 3.6, $f(x_n) \to L$ and $g(x_n) \to M$ as $n \to \infty$. By Theorem 2.12i, $f(x_n) + g(x_n) \to L + M$ as $n \to \infty$. Since this holds for any sequence $x_n \in I \setminus \{a\}$ which converges to a, we conclude by Theorem 3.6 that
$$\lim_{x\to a}(f+g)(x) = L + M = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$
The other rules follow in an analogous way from Theorem 2.12ii through 2.12iv.

Similarly, the Sequential Characterization of Limits can be combined with the Squeeze and Comparison Theorems for sequences to establish the following results.

3.9 Theorem. [Squeeze Theorem for Functions].
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined everywhere on I except possibly at a.
1. If $g(x) \leq h(x) \leq f(x)$ for all $x \in I \setminus \{a\}$, and
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L,$$
Then the limit of $h(x)$ exists, as $x \to a$, and
$$\lim_{x\to a} h(x) = L.$$
2. If $|g(x)| \leq M$ for all $x \in I \setminus \{a\}$ and $f(x) \to 0$ as $x \to a$, then
$$\lim_{x\to a} f(x)g(x) = 0.$$

3.10 Theorem. [Comparison Theorem for Functions].
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined everywhere on I except possibly at a. If f and g have a limit as x approaches a and $f(x) \leq g(x)$ for all $x \in I \setminus \{a\}$, then
$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$
We shall refer to this last result as taking the limit of an inequality.

The limit theorems (Theorems 3.8, 3.9, and 3.10) allow us to prove that limits exist without resorting to ϵ's and δ's.

3.11 Example.

Prove that

$$\lim_{x \to 1} \frac{x-1}{3x+1} = 0.$$

Proof.

By Example 3.2, $x - 1 \to 0$ and $3x + 1 \to 4$ as $x \to 1$. Hence, by Theorem 3.8, $\frac{x-1}{3x+1} \to \frac{0}{4} = 0$ as $x \to 1$.

3.2 One-sided Limits and Limits at Infinity

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3.12 Definition.

Let $a \in \mathbb{R}$ and f be a real function.

1. $f(x)$ is said to converge to L as x approaches a from the right if and only if f is defined on some open interval I with left endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I, and a) such that
   $$a + \delta \in I \text{ and } a < x < a + \delta \text{ imply } |f(x) - L| < \epsilon.$$
   In this case we call L the *right-hand limit* of f at a, and denote it by
   $$f(a+) \coloneqq L =: \lim_{x \to a^+} f(x).$$
2. $f(x)$ is said to converge to L as x approaches a from the left if and only if f is defined on some open interval I with right endpoint a and for every $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, I and a) such that $a - \delta \in I$ and $a - \delta < x < a$ imply $|f(x) - L| < \epsilon$. In this case we call L the left-hand limit of f at a and denote it by
   $$f(a-) \coloneqq L =: \lim_{x \to a^-} f(x).$$

3.13 Examples.

1. Prove that
   $$f(x) = \begin{cases} x + 1 & x \geq 0 \\ x - 1 & x < 0 \end{cases}$$
   Has one-sided limits at $a = 0$ but $\lim_{x \to 0} f(x)$ does not exist.
2. Prove that
   $$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Proof.

1. Let $\epsilon > 0$ and set $\delta = \epsilon$. If $0 < x < \delta$, then $|f(x) - 1| = |x| < \delta < \epsilon$. Hence $\lim_{x \to 0^+} f(x)$ exists and equals 1. Similarly, $\lim_{x \to 0^-} f(x)$ exists and equals -1. However, $x_n = \frac{(-1)^n}{n} \to 0$ but $f(x_n) = (-1)^n \left(1 + \frac{1}{n}\right)$ does not converge as $n \to \infty$. Hence by the Sequential Characterization of Limits, $\lim_{x \to 0} f(x)$ does not exist.
2. Let $\epsilon > 0$ and set $\delta = \epsilon^2$. If $0 < x < \delta$, then $|f(x)| = \sqrt{x} < \sqrt{\delta} = \epsilon$.

Not every function has one-sided limits (see Example 3.7). Examples 3.13 show that even when a function has one-sided limits, it may not have a two-sided limit. The following result, however, shows that if both one-sided limits, at a point a, exist and are EQUAL, then the two-sided limit at a exists.

3.14 Theorem.

Let f be a real function. Then the limit
$$\lim_{x \to a} f(x)$$
Exists and equals L if and only if
$$L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Proof.

If the limit L of $f(x)$ exists as $x \to a$, then given $\epsilon > 0$ choose $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. Since any x which satisfies $a < x < a + \delta$ or $a - \delta < x < a$ also satisfies $0 < |x - a| < \delta$, it is clear that both the left and right limits of $f(x)$ exist as $x \to a$ and satisfy (3). Conversely, suppose that (3) holds. Then given $\epsilon > 0$ there exists a $\delta_1 > 0$ (respectively, a $\delta_2 > 0$) such that $a < x < a + \delta_1$ (respectively, $a - \delta_2 < x < a$) implies $|f(x) - L| < \epsilon$. Set $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta$ implies $a < x < a + \delta_1$ or $a - \delta_2 < x < a$ (depending on whether x is to the right or to the left of a). Hence (1) holds; that is $f(x) \to L$ as $x \to a$.

3.15 Definition.

Let $a, L \in \mathbb{R}$ and let f be a real function.

1. $f(x)$ is said to converge to L as $x \to \infty$ if and only if there exists a $c > 0$ such that $(c, \infty) \subset \mathrm{Dom}(f)$ and given $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$, in which case we shall write
$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty.$$
Similarly, $f(x)$ is said to converge to L as $x \to -\infty$ if and only if there exists a $c > 0$ such that $(c, -\infty) \subset \mathrm{Dom}(f)$ and given $\epsilon > 0$ there is an $M \in \mathbb{R}$ such that $x < M$ implies $|f(x) - L| < \epsilon$, in which case we shall write
$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to \infty.$$
2. The function $f(x)$ is said to converge to ∞ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > M$, in which case we shall write
$$\lim_{x\to a} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to a.$$
Similarly, $f(x)$ is said to converge to $-\infty$ as $x \to a$ if and only if there is an open interval I containing a such that $I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < M$, in which case we shall write
$$\lim_{x\to a} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to a.$$

3.16 Examples.

1. Prove that $\frac{1}{x} \to 0$ as $x \to \infty$.
2. Prove that
$$\lim_{x\to 1-} f(x) := \lim_{x\to 1-} \frac{x+2}{2x^2 - 3x + 1} = -\infty.$$

Proof.

1. Given $\epsilon > 0$, set $M = \frac{1}{\epsilon}$. If $x > M$, then $\left|\frac{1}{x}\right| = \frac{1}{x} < \frac{1}{M} = \epsilon$. Thus $\frac{1}{x} \to 0$ as $x \to \infty$.
2. Let $M \in \mathbb{R}$. We must show that $f(x) < M$ for x near but to the left of 1 (no matter how large and negative M is). Without loss of generality, assume that $M < 0$. As x converges to 1 from the left, $2x^2 - 3x + 1$ is negative and converges to 0. (Observe that $2x^2 - 3x + 1$ is a parabola opening upward with roots $\frac{1}{2}$ and 1.) Therefore, choose $\delta \in (0, 1)$ such that $1 - \delta < x < 1$ implies $\frac{2}{M} < 2x^2 - 3x + 1 < 0$; that is, $-\frac{1}{2x^2-3x+1} > \left(-\frac{M}{2}\right) > 0$. Since $0 < x < 1$ also implies $2 < x + 2 < 3$, it follows that $-\frac{x+2}{2x^2-3x+1} > (-M)$; that is,
$$f(x) = \frac{x+2}{2x^2 - 3x + 1} < M$$
For all $1 - \delta < x < 1$.

In order to unify the presentation of one-sided, two-sided, and infinite limits, we introduce the following notation. Let a be an extended real number, and let I be a nondegenerate open interval which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. If a is finite and I contains a, then
$$\lim_{\substack{x\to a\\ x\in I}} f(x)$$
Will denote $\lim_{x\to a} f(x)$ (when it exists); if a is a finite left endpoint of I, then (4) will denote $\lim_{x\to a+} f(x)$ (when it exists); if a is a finite right endpoint of I, then (4) will denote $\lim_{x\to a-} f(x)$ (when it exists); if $a = \pm\infty$ is an endpoint of I, then (4) will denote $\lim_{x\to\pm\infty} f(x)$ (when each exists).
Using this notation, we can state a Sequential Characterization of Limits valid for two-sided, one-sided, and infinite limits.

3.17 Theorem.
Let a be an extended real number, and let I be a nondegenerate open interval which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. Then
$$\lim_{\substack{x\to a\\ x\in I}} f(x)$$
Exists and equals L if and only if $f(x_n) \to L$ for all sequences $x_n \in I$ which satisfy $x_n \neq a$ and

$x_n \to a$ as $n \to \infty$.

Proof.
Since we have already proved this for two-sided limits, we must show it for the remaining eight cases which notation (4) represents. Since the proofs are similar, we shall give the details for only one of these cases, namely the case when a belongs to I and $L = \infty$. Thus, we must prove that $f(x) \to \infty$ as $x \to a$ if and only if $f(x_n) \to \infty$ for any sequence $x_n \in I$ which converges to a and satisfies $x_n \neq a$ for $n \in \mathbb{N}$.
Suppose first that $f(x) \to \infty$ as $x \to a$. If $x_n \in I$, $x_n \to a$ as $n \to \infty$, and $x_n \neq a$, then given $M \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > M$, and there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - a| < \delta$. Consequently, $n \geq N$ implies $f(x_n) > M$; that is, $f(x_n) \to \infty$ as $n \to \infty$ as required.
Conversely, suppose to the contrary that $f(x_n) \to \infty$ for any sequence $x_n \in I$ which converges to a and satisfy $x_n \neq a$ but $f(x)$ does NOT converge to ∞ as $x \to a$. By the definition of "convergence" to ∞ there are numbers $M_0 \in \mathbb{R}$ and $x_n \in I$ such that $|x_n - a| < \frac{1}{n}$ and $f(x_n) \leq M_0$ for all $n \in \mathbb{N}$. The first condition implies $x_n \to a$ but the second condition implies that $f(x_n)$ does not converge to ∞ as $n \to \infty$. This contradiction proves Theorem 3.17 in the case $a \in I$ and $L = \infty$.

Using Theorem 3.17, we can prove limit theorems that are function analogues of Theorem 2.15 and Corollary 2.16. We leave this to the reader and will use these results as the need arises.

3.18 Example.
Prove that

$$\lim_{x\to\infty} \frac{2x^2 - 1}{1 - x^2} = -2.$$

Proof.
Since the limit of a product is the product of the limits, we have by Example 3.16i that $\frac{1}{x^m} \to 0$ as $x \to \infty$ for any $m \in \mathbb{N}$. Multiplying numerator and denominator of the expression above by $1/x^2$, we obtain

$$\lim_{x\to\infty} \frac{2x^2 - 1}{1 - x^2} = \lim_{x\to\infty} \frac{2 - \frac{1}{x^2}}{-1 + \frac{1}{x^2}} = \frac{\lim_{x\to\infty}\left(2 - \frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(-1 + \frac{1}{x^2}\right)} = \frac{2}{-1} = -2.$$

3.3 Continuity

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3.19 Definition.

Let E be nonempty subset of $\mathbb{R}$ and $f: E \to \mathbb{R}$.

1. f is said to be continuous at a point $a \in E$ if and only if given $\epsilon > 0$ there is a $\delta > 0$ (which in general depends on ϵ, f, and a) such that
   $$|x - a| < \delta \text{ and } x \in E \text{ imply } |f(x) - f(a)| < \epsilon.$$
2. f is said to be continuous on E (notation: $f: E \to \mathbb{R}$ is continuous) if and only if f is continuous at every $x \in E$.

3.20 Remark.

Let I be an open interval which contains a point a and $f: I \to \mathbb{R}$. Then f is continuous at $a \in I$ if and only if

$$f(a) = \lim_{x \to a} f(x).$$

Proof.

Suppose that $I = (c, d)$ and set $\delta_0 \coloneqq \min\{|c - a|, |d - a|\}$. If $\delta < \delta_0$, then $|x - a| < \delta$ implies $x \in I$. Therefore, condition (5) is identical to (1) when $f(a) = L, E = I$, and $\delta < \delta_0$. It follows that f is continuous at $a \in I$ if and only if $f(x) \to f(a)$ as $x \to a$.

3.21 Theorem.

Suppose that E is a nonempty subset of $\mathbb{R}$, that $a \in E$, and that $f: E \to \mathbb{R}$. Then the following statements are equivalent:

1. f is continuous at $a \in E$.
2. If x_n converges to a and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$.

3.22 Theorem.

Let E be a nonempty subset of $\mathbb{R}$ and $f, g: E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (respectively, continuous on the set E), then so are $f + g$, fg, and αf (for any $\alpha \in \mathbb{R}$). Moreover, $\frac{f}{g}$ is continuous at $a \in E$ when $g(a) \neq 0$ (respectively, on E when $g(x) \neq 0$ for all $x \in E$).

It follows from Exercise 3.1.6, 3.1.7, and 3.1.8 that if f, g are continuous at a point $a \in E$ or on a set E, then so are $|f|, f^+, f^-, f \vee g$, and $f \wedge g$. We also notice by Exercise 3.2.3 that every polynomial is continuous on $\mathbb{R}$.

3.23 Definition.

Suppose that A and B are subsets of $\mathbb{R}$, that $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$. If $f(A) \subseteq B$ for every $x \in A$, then the composition of g with f is the function $g \circ f: A \to \mathbb{R}$ defined by

$$(g \circ f)(x) \coloneqq g(f(x)), \qquad x \in A.$$

3.24 Theorem.

Suppose that A and B are subsets of $\mathbb{R}$, that $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$, and that $f(x) \in B$ for every $x \in A$.

1. If $A \coloneqq I \setminus \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints, if
   $$L \coloneqq \lim_{\substack{x \to a \\ x \in I}} f(x)$$
   Exists and belongs to B, and if g is continuous at $L \in B$, then
   $$\lim_{\substack{x \to a \\ x \in I}} (g \circ f)(x) = g\left(\lim_{\substack{x \to a \\ x \in I}} f(x)\right).$$
2. If f is continuous at $a \in A$ and g is continuous at $f(a) \in B$, then $g \circ f$ is continuous at $a \in A$.

Proof.

Suppose that $x_n \in I \setminus \{a\}$ and that $x_n \to a$ as $n \to \infty$. Since $f(A) \subseteq B, f(x_n) \in B$. Also, by the Sequential Characterization of Limits (Theorem 3.17), $f(x_n) \to L$ as $n \to \infty$. Since g is continuous at $L \in B$, it follows from Theorem 3.17, $g \circ f(x) \to g(L)$ as $x \to a$ in I. This proves i). A similar proof establishes part ii).

3.25 Definition.
Let E be a nonempty subset of $\mathbb{R}$. A function $f: E \to \mathbb{R}$ is said to be bounded on E if and only if there is an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$, in which case we shall say that f is dominated by M on E.

3.26 Theorem. [Extreme Value Theorem].
If I is a closed, bounded interval and $f: I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover, if

$$M = \sup_{x \in I} f(x) \quad \text{and} \quad m = \inf_{x \in I} f(x),$$

Then there exist points, $x_m, x_M \in I$ such that

$$f(x_M) = M \quad \text{and} \quad f(x_m) = m.$$

Proof.
Suppose first that f is not bounded on I. Then there exist $x_n \in I$ such that

$$|f(x_n)| > n, \qquad n \in \mathbb{N}.$$

Since I is bounded, we know (by the Bolzano-Weierstrass Theorem) that $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \to a$ as $k \to \infty$. Since I is closed, we also know (by the Comparison Theorem) that $a \in I$. In particular, $f(a) \in \mathbb{R}$. On the other hand, substituting n_k for n in (7) and taking the limit of this inequality as $k \to \infty$, we have $|f(a)| = \infty$, a contradiction. Hence, the function f is bounded on I.
We have proved that both M and m are finite real numbers. To show that there is an $x_M \in I$ such that $f(x_M) = M$, suppose to the contrary that $f(x) < M$ for all $x \in I$. Then the function

$$g(x) = \frac{1}{M - f(x)}$$

Is continuous, hence bounded on I. In particular, there is a $C > 0$ such that $|g(x)| = g(x) \leq C$. It follows that

$$f(x) \leq M - \frac{1}{C}$$

For all $x \in I$. Taking the supremum of (8) over all $x \in I$, we obtain $M \leq M - \frac{1}{C} < M$, a contradiction. Hence, there is an $x_M \in I$ such that $f(x_M) = M$. A similar argument proves that there is an $x_m \in I$ such that $f(x_m) = m$.

3.27 Remark.
The Extreme Value Theorem is false is either "closed" or "bounded" is dropped from the hypotheses.

Proof.
The interval $(0,1)$ is bounded but not closed, and the function $f(x) = \frac{1}{x}$ is continuous and unbounded on $(0,1)$. The interval $[0,\infty)$ is closed but not bounded, and the function $f(x) = x$ is continuous and unbounded on $[0,\infty)$.

3.28 Lemma.
Suppose that $a < b$ and that $f: [a,b) \to \mathbb{R}$. If f is continuous at a point $x_0 \in [a,b)$ and $f(x_0) > 0$, then there exist a positive number ϵ and a point $x_1 \in [a,b)$ such that $x_1 > x_0$ and $f(x) > \epsilon$ for all $x \in [x_0, x_1]$.

Strategy: The idea behind this proof is simple. If $f(x_0) > 0$, then $f(x) > \frac{f(x_0)}{2}$ for x near x_0. Here are the details.
Proof.

Let $\epsilon = \frac{f(x_0)}{2}$. Since $x_0 < b$, it is easy to see that $\delta_0 := \frac{b-x_0}{2}$ is positive and that $a \le x < x_0 + \delta_0$ implies $x \in [a, b)$. Use Definition 3.19 to choose $0 < \delta < \delta_0$ such that $x \in [a, b)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \epsilon$.

Fix $x_1 \in (x_0, x_0 + \delta)$ and suppose that $x \in [x_0, x_1]$. By the choice of ϵ and δ, it is clear that

$$-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}.$$

Solving the left-hand inequality for $f(x)$, we conclude that $f(x) > \frac{f(x_0)}{2} = \epsilon$, as promised.

A real number y_0 is said to *lie between* two numbers c and d if and only if $c < y_0 < d$ or $d < y_0 < c$.

3.29 Theorem. [Intermediate Value Theorem].

Suppose that $a < b$ and that $f: [a, b] \to \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

Proof.

We may suppose that $f(a) < y_0 < f(b)$. Consider the set $E = \{x \in [a, b]: f(x) < y_0\}$ (see Figure 3.3). Since $a \in E$ and $E \subseteq [a, b]$, E is a nonempty, bounded subset of $\mathbb{R}$. Hence, by the Completeness Axiom, $x_0 := \sup E$ is a finite real number. It remains to prove that $x_0 \in (a, b)$ and $f(x_0) = y_0$.

Choose by Theorem 2.11 a sequence $x_n \in E$ such that $x_n \to x_0$ as $n \to \infty$. Since $E \subseteq [a, b]$, it follows from Theorem 2.17 that $x_0 \in [a, b]$. Moreover, by the continuity of f and the definition of E, we have $f(x_0) = \lim_{n\to\infty} f(x_n) \le y_0$.

To show that $f(x_0) = y_0$, suppose to the contrary that $f(x_0) < y_0$. Then $y_0 - f(x)$ is a continuous function on the interval $[a, b)$ whose value at $x = x_0$ is positive. Hence, by Lemma 3.28, we can choose an ϵ and an $x_1 > x_0$ such that $y_0 - f(x_1) > \epsilon > 0$. In particular, $x_1 \in E$ and $x_1 > \sup E$, a contradiction.

We have shown that $x_0 \in [a, b]$ and $y_0 = f(x_0)$. In view of our opening assumption, $f(a) < y_0 < f(b)$, it follows that x_0 cannot equal a or b. We conclude that $x_0 \in (a, b)$.

Thus, if f is continuous on $[a, b]$ and $f(a) \le y_0 \le f(b)$, then there is an $x_0 \in [a, b]$ such that $f(x_0) = y_0$.

If f fails to be continuous at a point a, we say that f is discontinuous at a and call a a point of discontinuity of f. How badly can a function behave near a point of discontinuity? The following examples can be interpreted as answers to this question. (See also Exercise 9.6.9)

3.30 Example.

Prove that the function

$$f(x) = \begin{cases} \frac{|x|}{x} & x \ne 0 \\ 1 & x = 0 \end{cases}$$

Is continuous on $(-\infty, 0)$ and $[0, \infty)$, discontinuous at 0, and that both $f(0+)$ and $f(0-)$ exist.

Proof.

Since $f(x) = 1$ for $x \ge 0$, it is clear that $f(0+) = 1$ exists and $f(x) \to f(a)$ as $x \to a$ for any $a > 0$. In particular, f is continuous on $[0, \infty)$. Similarly, $f(0-) = -1$ and f is continuous on $(-\infty, 0)$. Finally, since $f(0+) \ne f(0-)$, the limit of $f(x)$ as $x \to 0$ does not exist by Theorem 3.14. Therefore, f is not continuous at 0.

3.31 Example.

Assuming that $\sin x$ is continuous on $\mathbb{R}$, prove that the function

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \ne 0 \\ 1 & x = 0 \end{cases}$$

Is continuous on $(-\infty, 0)$ and $(0, \infty)$, discontinuous at 0, and neither $f(0+)$ nor $f(0-)$ exists. (see Figure 3.1.)

Proof.
The function $\frac{1}{x}$ is continuous for $x \neq 0$ by Theorem 3.8. Hence, by Theorem 3.24, $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $(-\infty, 0)$ and $(0, \infty)$. To prove that $f(0+)$ does not exist, let $x_n = \frac{2}{(2n+1)\pi}$, and observe (see Appendix B) that $\sin\left(\frac{1}{x_n}\right) = (-1)^n, n \in \mathbb{N}$. Since $x_n \downarrow 0$ but $(-1)^n$ does not converge, it follows from Theorem 3.21 (the Sequential Characterization of Continuity) that $f(0+)$ does not exist. A similar argument proves that $f(0-)$ does not exist.

3.32 Example.
The Dirichlet function is defined on $\mathbb{R}$ by

$$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Prove that every point $x \in \mathbb{R}$ is a point of discontinuity of f. (Such functions are called nowhere continuous.)
Proof.
By Theorem 1.18 and Exercise 1.3.3 (Density of Rationals and Irrationals), given any $a \in \mathbb{R}$ and $\delta > 0$ we can choose $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{R} \setminus \mathbb{Q}$ such that $|x_i - a| < \delta$ for $i = 1, 2$. Since $f(x_1) = 1$ and $f(x_2) = 0$, f cannot be continuous at a.

3.33 Example.
Prove that the function

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \text{ (in reduced form)} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Is continuous at every irrational in the interval $(0, 1)$ but discontinuous at every rational in $(0, 1)$.

Proof.
Let a be a rational in $(0, 1)$ and suppose that f is continuous at a. If x_n is a sequence of irrationals which converges to a, then $f(x_n) \to f(a)$; that is, $f(a) = 0$. But $f(a) \neq 0$ by definition. Hence, f is discontinuous at every rational in $(0, 1)$.
Let a be an irrational in $(0, 1)$. We must show that $f(x_n) \to f(a)$ for every sequence $x_n \in (0, 1)$ which satisfies $x_n \to a$ as $n \to \infty$. We may suppose that $x_n \in \mathbb{Q}$. For each $n \in \mathbb{N}$, write $x_n = \frac{p_n}{q_n}$ in reduced form. Since $f(a) = 0$, it suffices to show that $q_n \to \infty$ as $n \to \infty$. Suppose to the contrary that there exist integers $n_1 < n_2 < \cdots$ such that $|q_{n_k}| \leq M < \infty$ for $k \in \mathbb{N}$. Since $x_{n_k} \in (0, 1)$, it follows that the set

$$E := \left\{ x_{n_k} = \frac{p_{n_k}}{q_{n_k}} : k \in \mathbb{N} \right\}$$

Contains only a finite number of points. Hence, the limit of any sequence in E must belong to E, a contradiction since a is such a limit and is irrational.

To see how counterintuitive Example 3.33 is, try to draw a graph of $y = f(x)$. Strager things can happen.

3.34 Remark.
The composition of two functions $g \circ f$ can be nowhere continuous, even though f is discontinuous only on $\mathbb{Q}$ and g is discontinuous at only one point.

Proof.
Let f be the function given in Example 3.33 and set

$$g(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Clearly,

$$(g \circ f)(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Hence, $g \circ f$ is the Dirichlet function, nowhere continuous by Example 3.32.

In view of Example 3.33 and Remark 3.34, we must be skeptical of proofs which rely exclusively on geometric intuition. And although we shall use geometric intuition to suggest methods of proof for many results in subsequent chapters, these suggestions will always be followed by a careful rigorous proof which contains no fuzzy reasoning based on pictures or sketches no matter how plausible they seem.

3.4 Uniform Continuity

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3.35 Definition.
Let E be a nonempty subset of $\mathbb{R}$ and $f: E \to \mathbb{R}$. Then f is said to be uniformly continuous on E (notation: $f: E \to \mathbb{R}$ is uniformly continuous) if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \text{and} \quad x, a \in E \quad \text{imply} \quad |f(x) - f(a)| < \epsilon.$$

Notice that the δ in Definition 3.35 depends on ϵ and f, but not on a and x. This issue needs to be addressed when we prove that a given function is uniformly continuous on a specific set (e.g., by determining δ before a is mentioned).

3.36 Example.
Prove that $f(x) = x^2$ is uniformly continuous on the interval $(0, 1)$.

Proof.
Given $\epsilon > 0$, set $\delta = \frac{\epsilon}{2}$. If $x, a \in (0, 1)$, then $|x + a| \le |x| + |a| \le 2$. Therefore, if $x, a \in (0, 1)$ and $|x - a| < \delta$, then

$$|f(x) - f(a)| = |x^2 - a^2| = |x - a||x + a| \le 2|x - a| < 2\delta = \epsilon.$$

The definitions of continuity and uniform continuity are very similar. In fact, the only difference is that for a continuous function, the parameter δ may depend on a, whereas for a uniformly continuous function, δ must be chosen independently of a. In particular, every function uniformly continuous on E is also continuous on E. The following example shows that the converse of this statement is false unless some restriction is made of E.

3.37 Example.
Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof.
Suppose to the contrary that f is uniformly continuous on $\mathbb{R}$. Then there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < 1$ for all $x, a \in \mathbb{R}$. By the Archimedean Principle, choose $n \in \mathbb{N}$ so large that $n\delta > 1$. Set $a = n$ and $x = n + \frac{\delta}{2}$. Then $|x - a| < \delta$ and

$$1 > |f(x) - f(a)| = |x^2 - a^2| = n\delta + \frac{\delta^2}{4} > n\delta > 1.$$

The contradiction proves that f is not uniformly continuous on $\mathbb{R}$.

3.38 Lemma.
Suppose that $E \subseteq \mathbb{R}$ and that $f: E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, then $f(x_n)$ is Cauchy.

Proof.
Let $\epsilon > 0$ and choose $\delta > 0$ such that (9) holds. Since $\{x_n\}$ is Cauchy, choose $N \in \mathbb{N}$ such that $n, m \ge N$ implies $|x_n - x_m| < \delta$. Then $n, m \ge N$ implies $|f(x_n) - f(x_m)| < \epsilon$.

Notice that $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$ and $x_n = \frac{1}{n}$ is Cauchy but $f(x_n)$ is not. In particualr, $\frac{1}{x}$ is continuous but not uniformly continuous on the open interval $(0, 1)$. Notice how the graph of $y = \frac{1}{x}$ corroborates this fact. Indeed, as a gets closer to 0, the value of δ gets smaller (compare δ_1 to δ_0 in Figure 3.4) and hence cannot be chosen independently of a.

Thus on an open interval, continuity and uniform continuity are different, even if the interval is bounded. The following result shows that this is not the case for closed, bounded intervals. (This result is extremely important because uniform continuity is so strong. Indeed, we shall use it dozens of times before this book is finished.)

3.39 Theorem.
Suppose that I is a closed, bounded interval. If $f: I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

Proof.
Suppose to the contrary that f is continuous but not uniformly continuous on I. Then there is an $\epsilon_0 > 0$ and points $x_n, y_n \in I$ such that $|x_n - y_n| < \frac{1}{n}$ and

$$|f(x_n) - f(y_n)| \geq \epsilon_0, \qquad n \in \mathbb{N}.$$

By the Bolzano-Weierstrass Theorem and the Comparison Theorem, the sequence $\{x_n\}$ has a subsequence, say x_{n_k}, which converges, as $k \to \infty$, to some $x \in I$. Similarly, the sequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ has a convergent subsequence, say $y_{n_{k_j}}$, which converges, as $j \to \infty$, to some $y \in I$. Since $x_{n_{k_j}} \to x$ as $j \to \infty$ and f is continuous, it follows from (10) that $|f(x) - f(y)| \geq \epsilon_0$; that is $f(x) \neq f(y)$. But $|x_n - y_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$ so Theorem 2.9 (the Squeeze Theorem) implies $x = y$. Therefore, $f(x) = f(y)$, a contradiction.

3.40 Theorem.
Suppose that $a < b$ and that $f: (a, b) \to \mathbb{R}$. Then f is uniformly continuous on (a, b) if and only if f can be continuously extended to $[a, b]$; that is, if and only if there is a continuous function $g: [a, b] \to \mathbb{R}$ which satisfies

$$f(x) = g(x), \qquad x \in (a, b).$$

Proof.
Suppose that f is uniformly continuous on (a, b). Let $x_n \in (a, b)$ converge to b as $n \to \infty$. Then $\{x_n\}$ is Cauchy; hence, by Lemma 3.38, so is $\{f(x_n)\}$. In particular,

$$g(b) := \lim_{n\to\infty} f(x_n)$$

Exists. This value does not change if we use a different sequence to approximate b. Indeed, let $y_n \in (a, b)$ be another sequence which converges to b as $n \to \infty$. Given $\epsilon > 0$, choose $\delta > 0$ such that (9) holds for $E = (a, b)$. Since $x_n - y_n \to 0$, choose $N \in \mathbb{N}$ so that $n \geq N$ implies $|x_n - y_n| < \delta$. By (9), then, $|f(x_n) - f(y_n)| < \epsilon$ for all $n \geq N$. Taking the limit of this inequality as $n \to \infty$, we obtain

$$\left|\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n)\right| \leq \epsilon$$

For all $\epsilon > 0$. It follows from Theorem 1.9 that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n).$$

Thus, $g(b)$ is well defined. A similar argument defines $g(a)$.
Set $g(x) = f(x)$ for $x \in (a, b)$. Then g is defined on $[a, b]$, satisfies (11), and is continuous on $[a, b]$ by the Sequential Characterization of Limits. Thus, f can be "continuously extended" to g as required.
Conversely, suppose that there is a function g continuous on $[a, b]$ which satisfies (11). By Theorem 3.39, g is uniformly continuous on $[a, b]$; hence, g is uniformly continuous on (a, b). We conclude that f is uniformly continuous on (a, b).

Let f be continuous on a bounded, open, nondegenerate interval (a, b). Notice that f is continuously extendable to $[a, b]$ if and only if the one-sided limits of f exist at a and b. Indeed, when they exist, we can always define g at a and b to be the values of these limits. In particular, we can prove that f is uniformly continuous without using ϵ's and δ's.

3.41 Example.
Prove that $f(x) = \frac{x-1}{\log x}$ is uniformly continuous on $(0, 1)$.

Proof.
It is clear that $f(x) \to 0$ as $x \to 0^+$. Moreover, by L'Hospotal's Rule (see Theorem 4.27),

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} \frac{1}{\frac{1}{x}} = 1.$$

Hence f is continuously extendable to $[0, 1]$, so by Theorem 3.40, f is uniformly continuous on $(0, 1)$.

Chapter 4 Differentiability on $\mathbb{R}$

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4.1 The Derivative

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4.1 Definition.
A real function f is said to be differentiable at a point $a \in \mathbb{R}$ if and only if f is defined on some open interval I containing a and

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Exists. In this case $f'(a)$ is called the derivative of f at a.

If f is differentiable at each point in a set E, then f' is a function on E. This function is denoted several ways:

$$D_x f = \frac{df}{dx} = f^{(1)} = f'.$$

When $y = f(x)$, we shall also use the notation $\frac{dy}{dx}$ or y' for f'. Higher-order derivatives are defined recursively; that is, if $n \in \mathbb{N}$, then $f^{(n+1)}(a) := \left(f^{(n)}\right)'(a)$, provided these derivatives exist. Higher-order derivatives are also denoted several ways, including $D_x^n f$, $\frac{d^n f}{dx^n}$, $f^{(n)}$, and by $\frac{d^n y}{dx^n}$ and $y^{(n)}$ when $y = f(x)$. The second derivatives $f^{(2)}$ (respectively, $y^{(2)}$) are usually written as f'' (respectively, y''), and when they exist at some point a, we shall say that f is twice differentiable at a.
Here are two characterizations of differentiability which we shall use to study derivatives. The first one, which characterizes the derivative in terms of the "chord function"

$$F(x) := \frac{f(x) - f(a)}{x - a} \quad x \neq a,$$

4.2 Theorem.
A real function f is differentiable at some point $a \in \mathbb{R}$ if and only if there exist an open interval I and a function$F: I \to \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at a, and

$$f(x) = F(x)(x - a) + f(a)$$

Holds for all $x \in I$, in which case $F(a) = f'(a)$.
Proof.
Notice once and for all that for $x \in I \setminus \{a\}$, (2) and (3) are equivalent. Suppose that f is differentiable at a. Then f is defined on some open interval I containing a, and the limit in (1) exists. Define F on I by (2) if $x \neq a$, and by $F(a) := f'(a)$. Then (3) holds for all $x \in I$, and F is continuous at a by (2) since $f'(a)$ exists.
Conversely, if (3) holds, then (2) holds for all $x \in I$, $x \neq a$. Taking the limit of (2) as $x \to a$, bearing in mind that F is continuous at a, we conclude that $F(a) = f'(a)$.

The second characterization of differentiability, in terms of linear approximations [i.e., how well $f(a+h) - f(a)$ can be approximated by a straight line through the origin] will be used in Chapter 11 to define the derivative of a function of several variables.

4.3 Theorem.
A real function f is differentiable at a if and only if there is a function T of the form $T(x) := mx$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0.$$

Proof.
Suppose that f is differentiable, and set $m := f'(a)$. Then by (1),

$$\frac{f(a+h) - f(a) - T(h)}{h} = \frac{f(a+h) - f(a)}{h} - f'(a) \to 0$$

As $h \to 0$.
Conversely, if (4) holds for $T(x) := mx$ and $h \neq 0$, then

$$\frac{f(a+h)-f(a)}{h} = m + \frac{f(a+h)-f(a)-mh}{h}$$
$$= m + \frac{f(a+h)-f(a)-T(h)}{h}.$$

By (4), the limit of this last expression is m. It follows that $\frac{f(a+h)-f(a)}{h} \to m$, as $h \to 0$; that is, that $f'(a)$ exists and equals m.

4.4 Theorem.
If f is differentiable at a, then f is continuous at a.

Proof.
Suppose that f is differentiable at a. By Theorem 4.2, there is an open interval I and a function F, continuous at a, such that $f(x) = f(a) + F(x)(x-a)$ for all $x \in I$. Taking the limit of this last expression as $x \to a$, we see that

$$\lim_{x\to a} f(x) = f(a) + F(a) \cdot 0 = f(a).$$

In particular, $f(x) \to f(a)$ as $x \to a$; that is, f is continuous at a.
Thus any function which fails to be continuous at a cannot be differentiable at a. The following example shows that the converse of Theorem 4.4 is false.

4.5 Example.
Show that $f(x) = |x|$ is continuous at 0 but not differentiable there.

Proof.
Since $x \to 0$ implies $|x| \to 0$, f is continuous at 0. On the other hand, since $|h| = h$ when $h > 0$ and $|h| = -h$ when $h < 0$, we have

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = 1 \text{ and } \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = -1$$

Since a limit exists if and only if its one-sided limits exist and are equal (Theorem 3.14), it follows that the limit in (1) does not exist when $a = 0$ and $f(x) = |x|$. Therefore, f is not differentiable at 0.

4.6 Definition.
Let I be a nondegenerate interval.

1. A function $f: I \to \mathbb{R}$ is said to be differentiable on I if and only if
$$f_I'(a) := \lim_{\substack{x\to a \\ x\in I}} \frac{f(x)-f(a)}{x-a}$$
Exists and is finite for every $a \in I$.
2. f is said to be continuously differentiable on I if and only if f_I' exists and is continuous on I.

4.7 Example.
The function $f(x) = x^{\frac{3}{2}}$ is differentiable on $[0,\infty)$ and $f'(x) = \frac{3\sqrt{x}}{2}$ for all $x \in [0,\infty)$.

Proof.
By the Power Rule (see Exercise 4.2.7), $f'(x) = \frac{3\sqrt{x}}{2}$ for all $x \in (0,\infty)$. And by definition,

$$f'(0) = \lim_{h\to 0^+} \frac{h^{\frac{3}{2}} - 0}{h} = \lim_{h\to 0^+} \sqrt{h} = 0.$$

Here is notation widely used in conjunction with Definition 4.6. Let I be a nondegenerate interval. For each $n \in \mathbb{N}$, define the collection of functions $C^n(I)$ by

$$C^n(I) := \left\{f: f: I \to \mathbb{R} \text{ and } f^{(n)} \text{ exists and is continuous on } I\right\}.$$

We shall denote the collection of f which belong to $C^n(I)$ for all $n \in \mathbb{N}$ by $C^\infty(I)$. Notice that $C^1(I)$ is precisely the collection of real functions which are continuously differentiable on I. When dealing with specific intervals, we shall drop the outer set of parentheses; for example, we shall write $C^n[a,b]$ for $C^n([a,b])$.
By modifying the proof of Theorem 4.4, we can show that if f is differentiable on I, then f is continuous on I. Thus, $C^\infty(I) \subset C^m(I) \subset C^n(I)$ for all integers $m > n > 0$.

The following example shows that not every function which is differentiable on $\mathbb{R}$ belongs to $C^1(\mathbb{R})$.

4.8 Example.
The function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is differentiable on $\mathbb{R}$ but not continuously differentiable on any interval which contains the origin.

Proof.
By definition,

$$f'(0) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0 \quad \text{and} \quad f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

For $x \neq 0$. Thus f is differentiable on $\mathbb{R}$ but $\lim_{x\to 0} f'(x)$ does not exist. In particular, f' is not continuous on any interval which contains the origin.

4.9 Remark.
$f(x) = |x|$ is differentiable on $[0, 1]$ and on $[-1, 0]$ but not on $[-1, 1]$.

Proof.
Since $f(x) = x$ when $x > 0$ and $f(x) = -x$ when $x < 0$, it is clear that f is differentiable on $[-1, 0) \cup (0, 1]$ [with $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$]. By Example 4.5, f is not differentiable at $x = 0$. However,

$$f'_{[0,1]}(0) = \lim_{h\to 0^+} \frac{|h|}{h} = 1 \quad \text{and} \quad f'_{[-1,0]}(0) = \lim_{h\to 0^-} \frac{|h|}{h} = -1.$$

Therefore, f is differentiable on $[0, 1]$ and on $[-1, 0]$.

4.2 Differentiability Theorems

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4.10 Theorem.

Let f and g be real functions and $\alpha \in \mathbb{R}$. If f and g are differentiable at a, then $f+g, \alpha f, f \cdot g$, and [when $g(a) \neq 0$] $\frac{f}{g}$ are all differentiable at a. In fact,

$$(f+g)'(a) = f'(a) + g'(a),$$
$$(\alpha f)'(a) = \alpha f'(a),$$
$$(f \cdot g)'(a) = g(a)f'(a) + f(a)g'(a),$$
$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}.$$

Proof.

The proofs of these rules are similar. We provide the details only for (7). By adding and subtracting $f(a)g(x)$ in the numerator of the left side of the following expression, we can write

$$\frac{f(x)g(x) - f(a)g(a)}{x-a} = g(x)\frac{\big(f(x) - f(a)\big)}{x-a} + f(a)\frac{\big(g(x) - g(a)\big)}{x-a}.$$

This last expression is a product of functions. Since g is continuous (see Theorem 4.4), it follows from Definition 4.1 and Theorem 3.8 that

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} = g(a)f'(a) + f(a)g'(a).$$

Formula (5) is called the Sum Rule, (6) is sometimes called the Homogeneous Rule, (7) is called the Product Rule, and (8) is called the Quotient Rule.

4.11 Theorem. [Chain Rule].

Let f and g be real functions. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = g'\big(f(a)\big)f'(a).$$

Proof.

By Theorem 4.2, there exist open intervals I and J, and functions $F: I \to \mathbb{R}$, continuous at a, and $G: J \to \mathbb{R}$, continuous at $f(a)$, such that $F(a) = f'(a), G\big(g(a)\big) = g'\big(f(a)\big)$,

$$f(x) = F(x)(x-a) + f(a), \qquad x \in I,$$

And

$$g(y) = G(y)\big(y - f(a)\big) + g\big(f(a)\big), \qquad y \in J.$$

Since f is continuous at a, we may assume (by making I smaller if necessary) that $f(x) \in J$ for all $x \in I$.

Fix $x \in I$. Apply (11) to $y = f(x)$ and (10) to x to write

$$\begin{aligned}(g \circ f)(x) = g\big(f(x)\big) &= G\big(f(x)\big)\big(f(x) - f(a)\big) + g\big(f(a)\big)\\ &= G\big(f(x)\big)F(x)(x-a) + (g \circ f)(a).\end{aligned}$$

Set $H(x) = G\big(f(x)\big)F(x)$ for $x \in I$. Since F is continuous at a and G is continuous at $f(a)$, it is clear that H is continuous at a. Moreover,

$$H(a) = G\big(f(a)\big)F(a) = g'\big(f(a)\big)f'(a).$$

It follows from Theorem 4.2, therefore, that $(g \circ f)'(a) = g'\big(f(a)\big)f'(a)$.

4.3 The Mean Value Theorem

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4.12 Lemma. [Rolle's Theorem].
Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a, b]$, differentiable on (a, b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

Proof.
By the Extreme Value Theorem, f has a finite maximum M and a finite minimum m on $[a, b]$. If $M = m$, then f is constant on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$.
Suppose that $M \neq m$. Since $f(a) = f(b)$, f must assume one of the values M or m at some point $c \in (a, b)$. By symmetry, we may suppose that $f(c) = M$. [That is, if we can prove the theorem when $f(c) = M$, then a similar proof establishes the theorem when $f(c) = m$.] Since M is the maximum of f on $[a, b]$, we have

$$f(c+h) - f(c) \leq 0$$

For all h which satisfy $c + h \in (a, b)$. In the case $h > 0$ this implies

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0,$$

And in this case $h < 0$ this implies

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

It follows that $f'(c) = 0$.

Notice once and for all that the proof of Rolle's Theorem proves a well-known result: The extreme values of a differentiable function on an open interval occur at critical points (i.e., at points where f' is zero).

4.13 Remark.
The continuity hypothesis in Rolle's Theorem cannot be relaxed at even one point in $[a, b]$.

Proof.
The function

$$f(x) = \begin{cases} x & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

Is continuous on $[0, 1)$, differentiable on $(0, 1)$, and $f(0) = f(1) = 0$, but $f'(x)$ is never zero.

4.14 Remark.
The differentiability hypothesis in Rolle's Theorem cannot be relaxed at even one point in (a, b).

Proof.
The function $f(x) = |x|$ is continuous on $[-1, 1]$, differentiable on $(-1, 1) \setminus \{0\}$, and $f(-1) = f(1)$, but $f'(x)$ is never zero.

We shall use Rolle's Theorem to obtain several useful results. The first is a pair of "Mean Value Theorems."

4.15 Theorem.
Suppose that $a, b \in \mathbb{R}$ with $a < b$.

1. [Generalized Mean Value Theorem] If f, g are continuous on $[a, b]$ and differentiable on

(a, b), then there is a $c \in (a, b)$ such that

$$g'(c)\big(f(b) - f(a)\big) = f'(c)\big(g(b) - g(a)\big).$$

2. [Mean Value Theorem] If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Proof.

1. Set $h(x) = f(x)\big(g(b) - g(a)\big) - g(x)\big(f(b) - f(a)\big)$. Since $h'(x) = f'(x)\big(g(b) - g(a)\big) - g'(x)\big(f(b) - f(a)\big)$, it is clear that h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b)$. Thus, by Rolle's Theorem, $h'(c) = 0$ for some $c \in (a, b)$.
2. Set $g(x) = x$ and apply part 1). (For a geometric interpretation of this result, see the opening paragraph of this section and Figure 4.3.)

The Generalized Mean Value Theorem is also called Cauchy's Mean Value Theorem. It is usually essential when comparing derivatives of two functions simultaneously, using higher-order derivatives to approximate functions, and studying certain kinds of generalized derivatives (e.g., see Taylor's Formula and l'Hospital's Rule in the next section, and Remark 14.32).
The Mean Value Theorem is most often used to extract information about f from f' (see, e.g., Exercise 4.3.4, 4.3.5, and 4.3.9). Perhaps the best known result of this type is the criterion for deciding when a differentiable function increases. To prove this result, we begin with the following nonmenclature.

4.16 Definition.
Let E be a nonempty subset of $\mathbb{R}$ and $f: E \to \mathbb{R}$.

1. f is said to be increasing (respectively, strictly increasing) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2$ imply $f(x_1) \le f(x_2)$ [respectively, $f(x_1) < f(x_2)$].
2. f is said to be decreasing (respectively, strictly decreasing) on E if and only if $x_1, x_2 \in E$ and $x_1 < x_2$ imply $f(x_1) \ge f(x_2)$ [respectively, $f(x_1) > f(x_2)$].
3. f is said to be monotone (respectively, strictly monotone) on E if and only if f is either decreasing or increasing (respectively, either strictly decreasing or strictly increasing) on E.

4.17 Theorem.
Suppose that $a, b \in \mathbb{R}$, with $a < b$, that f is continuous on $[a, b]$, and that f is differentiable on (a, b).

1. If $f'(x) > 0$ [respectively, $f'(x) < 0$] for all $x \in (a, b)$, then f is strictly increasing (respectively, strictly decreasing) on $[a, b]$.
2. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
3. If g is continuous on $[a, b]$ and differentiable on (a, b), and if $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f - g$ is constant on $[a, b]$.

Proof.
Let $a \le x_1 < x_2 \le b$. By the Mean Value Theorem, there is a $c \in (a, b)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Thus, $f(x_2) > f(x_1)$ when $f'(c) > 0$ and $f(x_2) < f(x_1)$ when $f'(c) < 0$. This proves part 1).
To prove part 2), notice that if $f' = 0$, then by the proof of part 1), f is both increasing and decreasing, and hence constant on $[a, b]$. Finally, part 3) follows from part 2) applied to $f - g$.

Theorem 4.17i is a great result. It makes checking a differentiable function for monotonicity a routine activity. However, there are many nondifferentiable functions which are monotone. For example, the greatest integer function,

$$f(x) = [x] \coloneqq n, \qquad n \le x < n + 1, n \in \mathbb{Z},$$

Is increasing on $\mathbb{R}$ but not even continuous, much less differentiable.

How badly can these nondifferentiable, monotone functions behave? The following result shows that, just like the greatest integer function, any function which is monotone on an interval always has left and right limits (contrast with Examples 3.31 and 3.32). This is a function analogue of the Monotone Convergence Theorem.

4.18 Theorem.

Suppose that f is increasing on $[a, b]$.

1. If $c \in [a, b)$, then $f(c+)$ exists and $f(c) \leq f(c+)$.
2. If $c \in (a, b]$, then $f(c-)$ exists and $f(c-) \leq f(c)$.

Proof.
By symmetry it suffices to show that $f(c-)$ exists and satisfies $f(c-) \leq f(c)$ for any fixed $c \in (a, b]$. Set $E = f\big((a, c)\big)$ and $s = \sup E$. Since f is increasing, $f(c)$ is an upper bound of E. Hence, s is a finite real number which satisfies $s \leq f(c)$. Given $\epsilon > 0$, choose by the Approximation Property an $x_0 \in (a, c)$ such that $s - \epsilon < f(x_0) \leq s$. Since f is increasing,

$$s - \epsilon < f(x_0) \leq f(x) \leq s$$

For all $x_0 < x < c$. Therefore, $f(c-)$ exists and satisfies $f(c-) = s \leq f(c)$.

We have seen (Example 3.32) that a function can be nowhere continuous (i.e., can have uncountably many points of discontinuity). How many points of discontinuity can a monotone function have?

***4.19 Theorem**
If f is monotone on an interval I, then f has at most countably many points of discontinuity on I.

Proof.
Without loss of generality, we may suppose that f is increasing. Since the countable union of at most countable sets is at most countable (Theorem 1.42ii), it suffices to show that the set of points of discontinuity of f can be written as a countable union of at most countable sets. Since $\mathbb{R}$ is the union of closed intervals $[-n, n]$, $n \in \mathbb{N}$, we may suppose that I is a closed, bounded interval $[a, b]$.
Let E represent the set of points of discontinuity of f on (a, b). By Theorem 4.18, $f(x-) \leq f(x) \leq f(x+)$ for all $x \in (a, b)$. Thus, f is discontinuous at such an x if and only if $f(x+) - f(x-) > 0$. It follows that

$$E = \bigcup_{j=1}^{\infty} A_j,$$

Where for each $j \in \mathbb{N}$, $A_j := \left\{x \in (a, b): f(x+) - f(x-) \geq \frac{1}{j}\right\}$. We will complete the proof by showing that each A_j is finite.
Suppose to the contrary that A_{j_0} is infinite for some j_0. Set $y_0 := j_0\big(f(b) - f(a)\big)$ and observe that since f is finite valued on I, y_0 is a finite real number. On the other hand, since A_{j_0} is infinite, then by symmetry we may suppose that there exist $x_1 < x_2 < \cdots$ in $[a, b]$ such that $f(x_k+) - f(x_k-) \geq 1/j_0$ for $k \in \mathbb{N}$. Since f is monotone, it follows that

$$f(b) - f(a) \geq \sum_{k=1}^{n} \Big(f(x_k+) - f(x_k-)\Big) \geq \frac{n}{j_0};$$

That is, $y_0 = j_0\big(f(b) - f(a)\big) \geq n$ for all $n \in \mathbb{N}$. Taking the limit of this last inequality as $n \to \infty$, we see that $y_0 = +\infty$. With this contradiction, the proof of the theorem is complete.

4.20 Example.
Prove that $1 + x < e^x$ for all $x > 0$.

Proof.
Let $f(x) = e^x - x$, and observe that $f'(x) = e^x - 1 > 0$ for all $x > 0$. It follows from Theorem 4.17i that $f(x)$ is strictly increasing on $(0, \infty)$. Thus $e^x - x = f(x) > f(0) = 1$ for $x > 0$. In particular, $e^x > x + 1$ for $x > 0$.

4.21 Theorem. [Bernoulli's Inequality].
Let α be a positive real number. If $0 < \alpha \leq 1$, then $(1 + x)^\alpha \leq 1 + \alpha x$ for all $x \in [-1, \infty)$, and if $\alpha \geq 1$, then if $\alpha \geq 1$, then $(1 + x)^\alpha \geq 1 + \alpha x$ for all $x \in [-1, \infty)$.

Proof.
The proofs of these inequalities are similar. We present the details only for the case $0 < \alpha \leq 1$. Fix $x \geq -1$ and let $f(t) = t^\alpha$, $t \in [0, \infty)$. Since $f'(t) = \alpha t^{\alpha-1}$, it follows from the Mean Value Theorem (applied to $a = 1$ and $b = 1 + x$) that

$$f(1+x) - f(1) = \alpha x c^{\alpha-1}$$
For some c between 1 and $1+x$.
Case 1. $x > 0$. Then $c > 1$. Since $0 < \alpha \le 1$ implies $\alpha - 1 \le 0$, it follows that $c^{\alpha-1} \le 1$, hence $xc^{\alpha-1} \le x$. Therefore, we have by (12) that
$$(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1} \le f(1) + \alpha x = 1 + \alpha x$$
As required.
Case 2. $-1 \le x \le 0$. Then $c \le 1$ so $c^{\alpha-1} \ge 1$. But since $x \le 0$, it follows that $xc^{\alpha-1} \le x$ as before and we can repeat (13) to obtain the same conclusion.

***4.22 Example.**
Prove that the sequence $\left(1+\frac{1}{n}\right)^n$ is increasing, as $n \to \infty$, and its limit L satisfies $2 < L \le 3$. (The limit L turns out to be an irrational number, the natural base $e = 2.718281828459 \ldots$.)
Proof.
The sequence $\left(1+\frac{1}{n}\right)^n$ is increasing, since by Bernoulli's Inequality,
$$\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} \le \left(1+\frac{1}{n+1}\right).$$
To prove that this sequence is bounded above, observe by the Binomial Formula that
$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k.$$
Now,
$$\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n(n-1)\ldots(n-k+1)}{n^k}\cdot\frac{1}{k!} \le \frac{1}{k!} \le \frac{1}{2^{k-1}}$$
For all $k \in \mathbb{N}$. It follows from Exercise 1.4.4c that
$$2 = \left(1+\frac{1}{1}\right) < \left(1+\frac{1}{n}\right)^n \le 1 + 1 + \sum_{k=1}^{n-1}\frac{1}{2^k} = 3 - \frac{1}{2^{n-1}} < 3$$
For $n > 1$. Hence, by the Monotone Convergence Theorem, the limit L exists and satisfies $2 < L \le 3$.

The last result in this section shows that although a differentiable function might not be continuously differentiable, its derivative does satisfy an intermediate value theorem. (This result is sometimes called Darboux's Theorem.)

***4.23 Theorem. [Intermediate Value Theorem For Derivatives].**
Suppose that f is differentiable on $[a,b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a,b)$ such that $f'(x_0) = y_0$.

Strategy: Let $F(x) := f(x) - y_0 x$. We must find an $x_0 \in (a,b)$ such that $F'(x_0) := f'(x_0) - y_0 = 0$. Since local extrema of a differentiable function F occur only where the derivative of F is zero (e.g., see the proof of Rolle's Theorem), it suffice to show that F has a local extremum at some $x_0 \in (a,b)$.

Proof.
Suppose that y_0 lies between $f'(a)$ and $f'(b)$. By symmetry, we may suppose that $f'(a) < y_0 < f'(b)$. Set $F(x) = f(x) - y_0 x$ for $x \in [a,b]$, and observe that F is differentiable on $[a,b]$. Hence, by the Extreme Value Theorem, F has an absolute minimum, say $F(x_0)$, on $[a,b]$. Now $F'(a) = f'(a) - y_0 < 0$, so $F(a+h) - F(a) < 0$ for $h > 0$ sufficiently small. Hence $F(a)$ is NOT the absolute minimum of F on $[a,b]$. Similarly, $F(b)$ is not the absolute minimum of F on $[a,b]$. Hence, the absolute minimum $F(x_0)$ must occur on (a,b); that is, $x_0 \in (a,b)$ and $F'(x_0) = 0$.

4.4 Taylor's Theorem and L'Hospital's Rule

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4.24 Theorem. [Taylor's Formula].
Let $n \in \mathbb{N}$ and let a, b be extended real numbers with $a < b$. If $f: (a,b) \to \mathbb{R}$, and if $f^{(n+1)}$ exists on (a,b), then for each pair of points $x, x_0 \in (a,b)$ there is a number c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof.
Without loss of generality, suppose that $x_0 < x$. Define

$$F(t) := \frac{(x-t)^{n+1}}{(n+1)!} \quad \text{and} \quad G(t) := f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

For each $t \in (a,b)$, and observe that the theorem will be proved if we can show that there is a c between x and x_0 such that

$$G(x_0) = F(x_0) \cdot f^{(n+1)}(c).$$

This looks like a job for the Generalized Mean Value Theorem.
To verify that F and G satisfy the hypotheses of the Generalized Mean Value Theorem, notice that

$$\frac{d}{dt}\left(\frac{f^{(k)}(t)}{k!}(x-t)^k\right) = \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}$$

For $t \in (a,b)$ and $k \in \mathbb{N}$. Telescoping, we obtain

$$G'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

For $t \in (a,b)$. On the other hand, by the Chain Rule

$$F'(t) = -\frac{(x-t)^n}{n!}$$

For $t \in \mathbb{R}$. Thus F and G are differentiable on (x_0, x), continuous on $[x_0, x]$ and satisfy

$$\frac{G'(t)}{F'(t)} = f^{(n+1)}(t), \qquad t \neq x.$$

By the Generalized Mean Value Theorem, there is a number $c \in (x_0, x)$ such that

$$\big(F(x) - F(x_0)\big)\, G'(c) = \big(G(x) - G(x_0)\big)\, F'(c).$$

Since $F(x) = G(x) = 0$ and $x \neq c$, it follows that $-F(x_0)G'(c) = -G(x_0)F'(c)$; that is, $G(x_0) = F(x_0) \cdot \frac{G'(c)}{F'(c)}$. We conclude by (15) that (14) holds, as promised.

We shall use this result in Chapter 7 to show that most of the functions you've used in calculus classes before are very nearly polynomials themselves. To lay some ground work for these results, we introduce some additional notation.

Define $0! = 1$ and $f^{(0)}(x) = f(x)$, and notice that $f(x_0) = \frac{f^{(0)}(x_0)}{0!}$. We shall call

$$P_n^{f,x_0}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

The Taylor Polynomial of order n generated by f centered at x_0. Clearly, for each $f \in \mathcal{C}^\infty(a,b)$, Taylor's Formula gives us an estimate of how well Taylor polynomials approxiamte f. In fact, since Taylor's Formula implies

$$\left|f(x) - P_n^{f,x_0}(x)\right| \le \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}\right|,$$

For some c between x and x_0 and the fraction $\frac{1}{(n+1)!}$ gets smaller as n gets larger, we see that when the derivatives of f are bounded, the higher-order Taylor polynomials approximate f better than the lower-order ones do.

Let's look at two specific examples to see how this works out in practice.

4.25 Example.
Let $f(x) = e^x$ and $n \in \mathbb{N}$.

1. Find the Taylor polynomial $P_n := P_n^{f,0}$.
2. Prove that if $x \in [-1,1]$, then
$$\left|e^x - P_n(x)\right| \le \frac{3}{(n+1)!}.$$
3. Find an n so large that P_n approximates e^x on $[-1,1]$ to four decimal places.

Proof.

1. Since $f^{(k)}(x) = e^x$ for all $x \in \mathbb{R}$ and $k = 0, 1, \dots$, it is clear that $f^{(k)}(0) = 1$ for all $k \ge 0$; that is, that
$$P_n^{e^x,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$
2. Let $c, x \in [-1,1]$. Clearly, $|e^c| \le e^1 < 3$ and $|x^n| \le 1$ for all $n \in \mathbb{N}$. But if c lies between x and 0, then $c \in [-1,1]$. Thus it follows from (17) that $\left|e^x - P_n(x)\right| \le \frac{|e^c x^{n+1}|}{(n+1)!} < \frac{3}{(n+1)!}$.
3. To get four-place accuracy, we want $\left|e^x - P_n(x)\right| \le .00005$. By part 2., this will hold when $\frac{3}{(n+1)!} < 0.00005$; that is, when $(n+1)! \ge 60000$. According to my calculator, this occurs when $n + 1 \ge 9$, so set $n = 8$.

4.26 Example.
Let $f(x) = \sin x$ and $n \in \mathbb{N}$.

1. Find the Taylor polynomial $P_{2n+1} := P_{2n+1}^{f,0}$.
2. Prove that if $x \in [-1,1]$, then
$$\left|\sin x - P_{2n+1}(x)\right| \le \frac{1}{(2n+2)!}.$$
3. Find an n so large that P_{2n+1} approximates $\sin x$ on $[-1,1]$ to three decimal places.

Proof.

1. Observe that $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, and $f^{(4)}(x) = \sin x$, right back where we started from. Thus it is clear that $f^{(2k)}(x) = (-1)^k \sin x$ and $f^{(2k+1)}(x) = (-1)^k \cos x$ for $k = 0, 1, \dots$. It follows that $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$ for $k \ge 0$; that is, that
$$P_{2n+1}^{\sin x,0}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$
2. Let $c, x \in [-1,1]$. Clearly, $\left|f^{2n+2}(c)\right| \le 1$ and $\left|x^{2n+2}\right| \le 1^{2n+2} = 1$ for all $n \in \mathbb{N}$. Thus it follows from (17) that $\left|\sin x - P_{2n+1}(x)\right| \le \frac{1}{(2n+2)!}$.
3. To get three-place accuracy, we want $\left|\sin x - P_{2n+2}(x)\right| \le .0005$. By part 2., this will hold when $\frac{1}{(2n+2)!} < 0.0005$; that is, when $(2n+2)! \ge 2000$. According to my calculator, this occurs when $2n + 2 \ge 7$, so set $n = 3$.

This next result is a widely known technique for evaluating limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Since it involves using information about derivatives to draw conclusions about the functions themselves, it should come as no surprise that the proof uses the Meav Value Theorem. (Notice that our statement is general enough to include one-sided limits and limits at infinity.)

4.27 Theorem. [L'Hospital's Rule].
Let a be an extended real number and I be an open interval which either contains a or has a as an endpoint. Suppose that f and g are differentiable on $I \setminus \{a\}$ and that $g(x) \ne 0 \ne g'(x)$ for all $x \in I \setminus \{a\}$. Suppose further that
$$A := \lim_{\substack{x \to a \\ x \in I}} f(x) = \lim_{\substack{x \to a \\ x \in I}} g(x)$$
Is either 0 or ∞. If
$$B := \lim_{\substack{x \to a \\ x \in I}} \frac{f'(x)}{g'(x)}$$

Exists as an extended real number, then

$$\lim_{\substack{x\to a\\ x\in I}} \frac{f(x)}{g(x)} = \lim_{\substack{x\to a\\ x\in I}} \frac{f'(x)}{g'(x)}.$$

Proof.

Let $x_k \in I$ be distinct points with $x_k \to a$ as $k \to \infty$ such that either $x_k < a$ or $x_k > a$ for all $k \in \mathbb{N}$. By the Sequential Characterization of Limits and by the characterization of two-sided limits in terms of one-sided limits, it suffices to show that $\frac{f(x_k)}{g(x_k)} \to B$ as $k \to \infty$.

We suppose for simplicity that $B \in \mathbb{R}$. (For the cases $B = \pm\infty$, see Exercise 4.4.10.) Notice once and for all, since g' is never zero on I, that by Mean Value Theorem the differences $g(x) - g(y)$ are never zero for $x, y \in I, x \neq y$, provided either $x, y > a$ or $x, y < a$. Hence, we can divide by these differences at will.

Case 1.

$A = 0$ and $a \in \mathbb{R}$. Extend f and g to $I \cup \{a\}$ by $f(a) := 0 =: g(a)$. By hypothesis, f and g are continuous on $I \cup \{a\}$ and differentiable on $I \setminus \{a\}$. Hence by the Generalized Mean Value Theorem, there is a c_k between x_k and $y := a$ such that

$$\frac{f(x_k) - f(y)}{g(x_k) - g(y)} = \frac{f'(c_k)}{g'(c_k)}.$$

Since $f(y) = g(y) = 0$, it follows that

$$\frac{f(x_k)}{g(x_k)} = \frac{f(x_k) - f(y)}{g(x_k) - g(y)} = \frac{f'(c_k)}{g'(c_k)}.$$

Let $k \to \infty$. Since c_k lies between x_k and a, c_k also converges to a as $k \to \infty$. Hence hypothesis and (21) imply $\frac{f(x_k)}{g(x_k)} \to B$ as $k \to \infty$.

Case 2.

$A = \pm\infty$ and $a \in \mathbb{R}$. We suppose by symmetry that $A = +\infty$. For each $k, n \in \mathbb{N}$, apply the Generalized Mean Value Theorem to choose a $c_{k,n}$ between x_k and x_n such that (20) holds for x_n in place of y and $c_{k,n}$ in place of c_k. Thus

$$\frac{f(x_n)}{g(x_n)} - \frac{f(x_k)}{g(x_n)} = \frac{f(x_n) - f(x_k)}{g(x_n)} = \frac{1}{g(x_n)} \cdot \big(g(x_n) - g(x_k)\big) \cdot \frac{f'(c_{k,n})}{g'(c_{k,n})}$$
$$= \left(1 - \frac{g(x_k)}{g(x_n)}\right) \frac{f'(c_{k,n})}{g'(c_{k,n})};$$

That is,

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_k)}{g(x_n)} - \frac{g(x_k)}{g(x_n)} \cdot \frac{f'(c_{k,n})}{g'(c_{k,n})} + \frac{f'(c_{k,n})}{g'(c_{k,n})}.$$

Since $A = \infty$, it is clear that $\frac{1}{g(x_n)} \to 0$ as $n \to \infty$, and since $c_{k,n}$ lies between x_k and x_n, it is also clear that $c_{k,n} \to a$, as $k, n \to \infty$. Thus (22) and hypothesis should imply that $\frac{f(x_n)}{g(x_n)} \approx 0 - 0 + B = B$ for large n and k. Specifically, let $0 < \epsilon < 1$. Since $c_{k,n} \to a$ as $k, n \to \infty$, choose an N_0 so large that $n \geq N_0$ implies $\left|\frac{f'(c_{N_0,n})}{g'(c_{N_0,n})} - B\right| < \frac{\epsilon}{3}$. Since $g(x_n) \to \infty$, choose an $N > N_0$ such that $\left|\frac{f(x_{N_0})}{g(x_n)}\right|$ and the product $\left|\frac{g(x_{N_0})}{g(x_n)}\right| \cdot \left|\frac{f'(c_{N_0,n})}{g'(c_{N_0,n})}\right|$ are both less than $\frac{\epsilon}{3}$ for all $n \geq N$. It follows from (22) that for any $n \geq N$,

$$\left|\frac{f(x_n)}{g(x_n)} - B\right| \leq \left|\frac{f(x_{N_0})}{g(x_n)}\right| + \left|\frac{g(x_{N_0})}{g(x_n)} \frac{f'(c_{N_0,n})}{g'(c_{N_0,n})}\right| + \left|\frac{f'(c_{N_0,n})}{g'(c_{N_0,n})} - B\right| < \epsilon.$$

Hence, $\frac{f(x_n)}{g(x_n)} \to B$ as $n \to \infty$.

Case 3.

$a = \pm\infty$. We suppose by symmetry that $a = +\infty$. Choose $c > 0$ such that $I \supset (c, \infty)$. For each $y \in \left(0, \frac{1}{c}\right)$, set $\phi(y) = f\left(\frac{1}{y}\right)$ and $\psi(y) = g\left(\frac{1}{y}\right)$. Notice that by the Chain Rule,

$$\frac{\phi'(y)}{\psi'(y)} = \frac{f'\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)}{g'\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)} = \frac{f'\left(\frac{1}{y}\right)}{g'\left(\frac{1}{y}\right)}.$$

Thus, for $x = \frac{1}{y} \in (c, \infty)$, $\frac{f'(x)}{g'(x)} = \frac{\phi'(y)}{\psi'(y)}$. Since $x \to \infty$ if and only if $y = \frac{1}{x} \to 0^+$, it follows that ϕ and ψ satisfy the hypotheses of Cases 1 or 2 for $a = 0$ and $I = \left(0, \frac{1}{c}\right)$. In particular,

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{y\to 0^+} \frac{\phi'(y)}{\psi'(y)} = \lim_{y\to 0^+} \frac{\phi(y)}{\psi(y)} = \lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

L'Hospital Rule can be used to compare the relative rates of growth of two functions. For example, the next result shows that as $x \to \infty$, e^x converges to ∞ much faster than x^2 does.

4.28 Example.

Prove that $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$.

Proof.

Since the limit of $\frac{x^2}{e^x}$ and $\frac{x}{e^x}$ are of the form $\frac{\infty}{\infty}$, we apply l'Hospital's Rule twice to verify

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{x}{e^x} = 0.$$

For each subsequent application of l'Hospital's Rule, it is important to check that the hypotheses still hold. For example,

$$\lim_{x\to 0} x^2/(x^2 + \sin x) = \lim_{x\to 0} \frac{2x}{2x + \cos x} = 0 \neq 1 = \lim_{x\to 0} \frac{2}{2 - \sin x}.$$

Notice that the middle limit is not of the form $0/0$.

L'Hospital Rule can be used to evaluate limits of the form $0 \cdot \infty = -0(-\infty)$.

4.29 Example.

Find $\lim_{x\to 0+} x \log x$.

Solution.

By writing x as $1/(1/x)$, we see that the limit in question is of the form $\frac{\infty}{\infty}$. Hence, by L'Hospital's Rule,

$$\lim_{x\to 0+} x \log x = \lim_{x\to 0+} \frac{\log x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = 0.$$

The next two examples show that L'Hospital's Rule can also be used to evaluate limits of the form 1^∞ and 0^0.

4.30 Example.

Find $L = \lim_{x\to 0+} (1 + 3x)^{1/x}$.

Solution.

If the limit exists, then by a law of logarithms and the fact that $\log x$ is continuous, we have $\log L = \lim_{x\to 0+} \log \frac{1+3x}{x}$. Thus it follows from l'Hospital's Rule and the Chain Rule that

$$\log L = \lim_{x\to 0+} \frac{\log(1 + 3x)}{x} = \lim_{x\to 0+} \frac{\frac{3}{1+3x}}{1} = 3.$$

In particular, the limit exists by l'Hospital's Rule and $L = e^{\log L} = e^3$.

4.31 Example.

Find $L = \lim_{x\to 1+} (\log x)^{1-x}$.

Solution.

If the limit $L > 0$ exists, then $\log L = \lim_{x\to 1} (1 - x) \log(\log x)$ is of the form $0 \cdot \infty$. Hence, by l'Hospital's Rule,

$$\log L = \lim_{x \to 1} \frac{\log \log x}{1/(1-x)} = \lim_{x \to 1} \frac{1/\left(x \log x\right)}{1/(1-x)^2} = \lim_{x \to 1} \frac{-2(1-x)}{1 + \log x} = 0.$$

Therefore, the limit exists by l'Hospital's Rule and $L = e^0 = 1$.

4.5 Inverse Function Theorems

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4.32 Theorem.
Let I be a nondegenerate interval and suppose that $f: I \to \mathbb{R}$ is 1-1. If f is continuous on I, then $J := f(I)$ is an interval, f is strictly monotone on I, and f^{-1} is continuous and strictly monotone on J.

Proof.
Since f is 1-1 from I onto J, Theorem 1.30 implies that f^{-1} exists and takes J onto I.
To show that J is an interval, since I contains at least two points, so does J. Let $c, d \in J$ with $c < d$. By the definition of an interval, it suffices to prove that every $y_0 \in (c, d)$ belongs to J. Since f takes I onto J, there exist points $a, b \in I$ such that $f(a) = c$ and $f(b) = d$. Since y_0 lies between $f(a)$ and $f(b)$, we can use the Intermediate Value Theorem to choose an x_0 between a and b such that $y_0 = f(x_0)$. Since $x_0 \in I$ and f takes I onto J, $y_0 = f(x_0)$ must belong to J, as required.
Suppose that f is not strictly monotone on I. Then there exist points $a, b, c \in I$ such that $a < c < b$ but $f(c)$ does not lie between $f(a)$ and $f(b)$. Since f is 1-1, $f(a) \neq f(b)$, so by symmetry we may suppose that $f(a) < f(b)$. Since $f(c)$ does not lie between $f(a)$ and $f(b)$, it follows that either $f(c) < f(a) < f(b)$ or $f(a) < f(b) < f(c)$. Hence by the Intermediate Value Theorem, there is an $x_1 \in (a, b)$ such that either $f(x_1) = f(a)$ or $f(x_1) = f(b)$. Since f is 1-1, we conclude that either $x_1 = a$ or $x_1 = b$, both contradictions. Therefore, f is strictly monotone on I.
By symmetry, suppose that f is strictly increasing on I. To prove that f^{-1} is strictly increasing on J, suppose to the contrary that there exist $y_1, y_2 \in J$ such that $y_1 < y_2$ but $f^{-1}(y_1) \geq f^{-1}(y_2)$. Then $x_1 := f^{-1}(y_1)$ and $x_2 := f^{-1}(y_2)$ satisfy $x_1 \geq x_2$ and $x_1, x_2 \in I$. Since f is strictly increasing on I, it follows that $y_1 = f(x_1) \geq f(x_2) = y_2$, a contradiction. Thus, f^{-1} is strictly increasing on J.
It remains to prove that f^{-1} is continuous from the left and from the right at each $y_0 \in J$. We will provide the details for continuity from the right. To this end, suppose that f^{-1} is not continuous from the right at some $y_0 \in J$; that is, that there exist $y_n \in J$ such that $y_n > y_0$, $y_n \to y_0$ as $n \to \infty$, but that

$$f^{-1}(y_n) \geq a_0 > f^{-1}(y_0)$$

For some number a_0. Since I is an interval and f^{-1} takes J onto I, it follows that a_0 belongs to I and there is a $b_0 \in J$ such that $a_0 = f^{-1}(b_0)$. Substituting this into (23), we see that $f^{-1}(y_n) \geq f^{-1}(b_0) > f^{-1}(y_0)$. Since f is strictly increasing, we conclude that $y_n \geq b_0 > y_0$; that is, y_n cannot converge to y_0, a contradiction. A small argument verifies that f^{-1} is continuous from the left at each $y_0 \in J$. Thus f^{-1} is continuous on J.

4.33 Theorem. [Inverse Function Theorem].
Let I be an open interval and $f: I \to \mathbb{R}$ be 1-1 and continuous. If $b = f(a)$ for some $a \in I$ and if $f'(a)$ exists and is nonzero, then f^{-1} is differentiable at b and $(f^{-1})'(b) = \frac{1}{f'(a)}$

Proof.
By Theorem 4.32, f is strictly monotone, say strictly increasing on I, and f^{-1} exists and is both continuous and strictly increasing on the range $f(I)$. Moreover, since $a := f^{-1}(b) \in I$ and I is open, we can choose $c, d \in \mathbb{R}$ such that $a \in (c, d) \subset I$.
Let E_0 be the range of f on (c, d); that is, $E_0 = f((c, d))$. By Theorem 4.32, E_0 must be an interval. Since f is strictly increasing, it follows that $E_0 = (f(c), f(d))$. Hence, we can choose $\delta > 0$ so small that $0 < |h| < \delta$ implies $b + h \in E_0$. In particular, $f^{-1}(b + h)$ is defined for all $0 < |h| < \delta$.
Fix such an h and set $x = f^{-1}(b + h)$. Observe that $f(x) - f(a) = b + h - b = h$. Since f^{-1} is continuous, $x \to a$ if and only if $h \to 0$. Therefore, by direct substitution, we conclude that

$$\lim_{h \to 0} \frac{f^{-1}(b + h) - f^{-1}(b)}{h} = \lim_{x \to a} \frac{x - a}{f(x) - f(a)} = \frac{1}{f'(a)}.$$

This theorem is usually presented in elementary calculus texts in a form more easily remembered:
If $y = f(x)$ and $x = f^{-1}(y)$, then

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Notice that, by using this formula, we do not need to solve explicitly for f^{-1} to be able to compute $(f^{-1})'$.

4.34 Example.
If $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$, prove that $f^{-1}(x)$ exists at $x = 6$ and find a value for $(f^{-1})'(6)$.

Solution.
Observe that $f(1) = 6$ and $f'(x) > 0$ for all $x > 0$. Thus f is strictly increasing on $(0, \infty)$, and hence 1-1 there.
Let $I = (0, 2)$, $a = 1$, and $b = 6$. Then $f(a) = b$ and $f'(a) = 15 \neq 0$. Hence, it follows from the Inverse Function Theorem that $(f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{15}$.

Chapter 5 Integrability on $\mathbb{R}$

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5.1 The Riemann Integral

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5.1 Definition.
Let $a, b \in \mathbb{R}$ with $a < b$.

1. A partition of the interval $[a, b]$ is set of points $P = \{x_0, x_1, \dots, x_n\}$ such that
$$a = x_0 < x_1 < \cdots < x_n = b.$$
2. The norm of a partition $P = \{x_0, x_1, \dots, x_n\}$ is the number
$$||P|| = \max_{1 \le j \le n} |x_j - x_{j-1}|.$$
3. A refinement of a partition $P = \{x_0, x_1, \dots, x_n\}$ is a partition Q of $[a, b]$ which satisfies $Q \supseteq P$. In this case we say that Q is finer than P.

5.2 Example. [The Dyadic Partition].
Prove that for each $n \in \mathbb{N}$, $P_n = \left\{\frac{j}{2^n} : j = 0,1, \dots, 2^n\right\}$ is a partition of the interval $[0,1]$, and P_m is finer than P_n when $m > n$.

Proof.
Fix $n \in \mathbb{N}$. If $x_j = \frac{j}{2^n}$, then $0 = x_0 < x_1 < \cdots < x_{2^n} = 1$. Thus, P_n is a partition of $[0,1]$. Let $m > n$ and set $p = m - n$. If $0 \le j \le 2^n$, then $\frac{j}{2^n} = \frac{j2^p}{2^m}$ and $0 \le j2^p \le 2^m$. Thus P_m is finer than P_n.

5.3 Definition.
Let $a, b \in \mathbb{R}$ with $a < b$, let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of the interval $[a, b]$, set $\Delta x_j \coloneqq x_j - x_{j-1}$ for $j = 1,2, \dots, n$, and suppose that $f: [a, b] \to \mathbb{R}$ is bounded.

1. The upper Riemann sum of f over P is the number
$$U(f, P) \coloneqq \sum_{j=1}^{n} M_j(f) \Delta x_j ,$$
Where
$$M_j(f) \coloneqq \sup f\left(\left[x_{j-1}, x_j\right]\right) \coloneqq \sup_{t \in \left[x_{j-1}, x_j\right]} f(t) .$$
2. The lower Riemann sum of f over P is the number
$$L(f, P) \coloneqq \sum_{j=1}^{n} m_j(f) \Delta x_j ,$$
Where
$$m_j(f) \coloneqq \inf f\left(\left[x_{j-1}, x_j\right]\right) \coloneqq \inf_{t \in \left[x_{j-1}, x_j\right]} f(t) .$$

(**Note**: Since we assumed that f is bounded, the numbers $M_j(f)$ and $m_j(f)$ exist and are finite.)

5.4 Remark.
If $g: \mathbb{N} \to \mathbb{R}$, then
$$\sum_{k=m}^{n} \left(g(k+1) - g(k)\right) = g(n+1) - g(m)$$
For all $n \ge m$ in $\mathbb{N}$.

Proof.
The proof is by induction on n. The formula holds for $n = m$. If it holds for some $n - 1 \ge m$, then
$$\sum_{k=m}^{n} \left(g(k+1) - g(k)\right) = \left(g(n) - g(m)\right) + \left(g(n+1) - g(n)\right) = g(n+1) - g(m).$$

We shall refer to this algebraic identity by saying the sum telescopes to $g(n+1) - g(m)$. In

particular, if $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,b]$, the sum $\sum_{j=1}^{n} \Delta x_j$ telescopes to $x_n - x_0 = b - a$.
Before we define what it means for a function to be integrable, we make several elementary observations concerning upper and lower sums.

5.5 Remark.
If $f(x) = \alpha$ is constant on $[a,b]$, then
$$U(f,P) = L(f,P) = \alpha(b-a)$$
For all partitions P of $[a,b]$.
Proof.
Since $M_j(f) = m_j(f) = \alpha$ for all j, the sum $U(f,P)$ and $L(f,P)$ telescopes to $\alpha(b-a)$

5.6 Remark.
$L(f,P) \le U(f,P)$ for all partitions P and all bounded functions f.
Proof.
By definition, $m_j(f) \le M_j(f)$ for all j.

The next result shows that as the partitions get finer, the upper and lower Riemann sums get nearer each other.

5.7 Remark.
If P is any partition of $[a,b]$ and Q is a refinement of P, then
$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$
Proof.
Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Since Q is finer than P, Q can be obtained from P in a finite number of steps by adding one point at a time. Hence it suffices to prove the inequalities above for the special case $Q = \{c\} \cup P$ for some $c \in (a,b)$. Moreover, by symmetry and Remark 5.6, we need only show $U(f,Q) \le U(f,P)$.
We may suppose that $c \notin P$. Hence, there is a unique index j_0 such that $x_{j_0-1} < c < x_{j_0}$. By definition, it is clear that
$$U(f,Q) - U(f,P) = M^{(\ell)}\left(c - x_{j_0-1}\right) + M^{(r)}\left(x_{j_0} - c\right) - M\Delta x_{j_0},$$
Where
$$M^{(\ell)} = \sup f\left(\left[x_{j_0-1}, c\right]\right), \qquad M^{(r)} = \sup f\left(\left[c, x_{j_0}\right]\right), \qquad \text{and}$$
$$M = \sup f\left(\left[x_{j_0-1}, x_{j_0}\right]\right).$$
By the Monotone Property of Suprema, $M^{(\ell)}$ and $M^{(r)}$ are both less than or equal to M. Therefore,
$$U(f,Q) - U(f,P) \le M\left(c - x_{j_0-1}\right) + M\left(x_{j_0} - c\right) - M\Delta x_{j_0} = 0.$$
5.8 Remark.
If P and Q are any partitions of $[a,b]$, then
$$L(f,P) \le U(f,Q).$$
Proof.
Since $P \cup Q$ is a refinement of P and Q, it follows from Remark 5.7 that
$$L(f,P) \le L(f,P \cup Q) \le U(f,P \cup Q) \le U(f,Q)$$
For any pair of partitions P, Q, whether Q is a refinement of P or not.

5.9 Definition.
Let $a, b \in \mathbb{R}$ with $a < b$. A function $f: [a,b] \to \mathbb{R}$ is said to be (*Riemann*) *integrable* on $[a,b]$ if and only if f is bounded on $[a,b]$, and for every $\epsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

Notice that this definition makes sense whether or not f is nonnegative. The connection between nonnegative functions and area was only a convenient vehicle to motivate Definition 5.9. Also notice that, by Remark 5.6, $U(f,P) - L(f,P) = |U(f,P) - L(f,P)|$ for all partitions P. Hence, $U(f,P) - L(f,P) < \epsilon$ is equivalent to $|U(f,P) - L(f,P)| < \epsilon$.
This section provides a good illustration of how mathematics works. The connection between area and integration leads directly to Definition 5.9. This definition between area and

integration leads directly to Definition 5.9. This definition, however, is not easy to apply in concrete situations. Thus, we search for conditions which imply integrability and are easy to apply. In view of Figure 5.2, it seems reasonable that a function is integrable if its graph does not jump around too much (so that it can be covered by thinner and thinner rectangles). Since the graph of a continuous function does not jump at all, we are led to the following simple criterion that is sufficient (but not necessary) for integrability.

5.10 Theorem.
Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on the interval $[a, b]$, then f is integrable on $[a, b]$.

Proof.
Let $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, choose $\delta > 0$ such that

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \frac{\epsilon}{b - a}.$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$ which satisfies $||P|| < \delta$. Fix an index j and notice, by the Extreme Value Theorem, that there are points x_m and x_M in $[x_{j-1}, x_j]$ such that

$$f(x_m) = m_j(f) \quad \text{and} \quad f(x_M) = M_j(f).$$

Since $||P|| < \delta$, we also have $|x_M - x_m| < \delta$. Hence by (1), $M_j(f) - m_j(f) < \frac{\epsilon}{b-a}$. In particular,

$$U(f, P) - L(f, P) = \sum_{j=1}^{n} \left(M_j(f) - m_j(f)\right) \Delta x_j < \frac{\epsilon}{b - a} \sum_{j=1}^{n} \Delta \mathrm{x}_j = \epsilon.$$

(The last step comes from telescoping.)

5.11 Example.
The Dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Is not Riemann integrable on [0,1].

Proof.
Clearly, f is bounded on [0,1]. By Theorem 1.18 and Exercise 1.3.3 (Density of Rationals and Irrationals), the supremum of f over any nondegenerate interval is 1, and the infimum of f over any nondegenerate interval is 0. Therefore, $U(f, P) - L(f, P) = 1 - 0 = 1$ for any partition P of the interval [0,1]; that is, f is not integrable on [0,1].

5.12 Example.
The function

$$f(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

Is integrable on [0,1].

Proof.
Let $\epsilon > 0$. Choose $0 < x_1 < 0.5 < x_2 < 1$ such that $x_2 - x_1 < \epsilon$. The set

$$P \coloneqq \{0, x_1, x_2, 1\}$$

Is a partition of [0,1]. Since $m_1(f) = 0 = M_1(f), m_2(f) = 0 < 1 = M_2(f)$, and $m_3(f) = 1 = M_3(f)$, it is easy to see that $U(f, P) - L(f, P) = x_2 - x_1 < \epsilon$. Therefore, f is integrable on [0,1].

We have defined integrability, but not the value of the integral. We remedy this situation by using the Riemann sums $U(f, P)$ and $L(f, P)$ to define upper and lower integrals.

5.13 Definition.
Let $a, b \in \mathbb{R}$ with $a < b$, and $f: [a, b] \to \mathbb{R}$ be bounded.

1. The upper integral of f on $[a, b]$ is the number

$$(U)\int_a^b f(x)\,dx := \inf\{U(f,P): P \text{ is a partition of } [a,b]\}.$$

2. The lower integral of f on $[a,b]$ is the number

$$(L)\int_a^b f(x)\,dx := \sup\{L(f,P): P \text{ is a partition of } [a,b]\}$$

3. If the upper and lower integrals of f on $[a,b]$ are equal, we define the integral of f on $[a,b]$ to be the common value

$$\int_a^b f(x)\,dx := (U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx.$$

This defines integration over nondegenerate intervals. Motivated by the interpretation of integration as area, we define the integral of any bounded function f on $[a,a]$ to be zero; that is,

$$\int_a^a f(x)\,dx := 0.$$

Although a bounded function might not be integrable (see Example 5.11 above), the following result shows that the upper and lower integrals of a bounded function always exist.

5.14 Remark.
If $f:[a,b]\to\mathbb{R}$ is bounded, then its upper and lower integrals exist and are finite, and satisfy

$$(L)\int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx.$$

Proof.
By Remark 5.8, $L(f,P) \le U(f,Q)$ for all partitions P and Q of $[a,b]$. Taking the supremum of this inequality over all partitions P of $[a,b]$, we have

$$(L)\int_a^b f(x)\,dx \le (U)(f,Q);$$

That is, the lower integral exists and is finite. Taking the infimum of this last inequality over all partitions Q of $[a,b]$, we conclude that the upper integral is also finite and greater than or equal to the lower integral.

5.15 Theorem.
Let $a,b\in\mathbb{R}$ with $a<b$, and $f:[a,b]\to\mathbb{R}$ be bounded. Then f is integrable on $[a,b]$ if and only if

$$(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx.$$

Proof.
Suppose that f is integrable. Let $\epsilon>0$ and choose a partition P of $[a,b]$ such that

$$U(f,P)-L(f,P)<\epsilon.$$

By definition, $(U)\int_a^b f(x)\,dx \le U(f,P)$ and the opposite inequality holds for the lower integral and the lower sum $L(f,P)$. Therefore, it follows from Remark 5.14 and (3) that

$$\left|(U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx\right| = (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx$$
$$\le U(f,P)-L(f,P)<\epsilon$$

Since it is valid for all $\epsilon>0$, (2) holds as promised.
Conversely, suppose that (2) holds. Let $\epsilon>0$ and choose, by the Approximation Property, partitions P_1 and P_2 of $[a,b]$ such that

$$U(f,P_1) < (U)\int_a^b f(x)\,dx + \frac{\epsilon}{2}$$

And

$$L(f,P_2) > (L)\int_a^b f(x)\,dx - \frac{\epsilon}{2}.$$

Set $P=P_1\cup P_2$. Since P is a refinement of both P_1 and P_2, it follows from Remark 5.7, the choices of P_1 and P_2, and (2) that

$$U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_2)$$
$$\le (U)\int_a^b f(x)\,dx + \frac{\epsilon}{2} - (L)\int_a^b f(x)\,dx + \frac{\epsilon}{2} = \epsilon$$

Since the integral has been defined only on intervals $[a,b]$, we have tacitly assumed that $a \le b$. We shall use the convention

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

To extend the integral to the case $a > b$. In particular, if $f(x)$ is integrable and nonpositive on $[a,b]$, then the area of the region bounded by the curves $y = f(x), y = 0, x = a$, and $x = b$ is given by $\int_b^a f(x)\,dx$.

5.16 Theorem.

If $f(x) = \alpha$ is constant on $[a,b]$, then

$$\int_a^b f(x)\,dx = \alpha(b-a).$$

Proof.

By Theorem 5.10, f is integrable on $[a,b]$. Hence, it follows from Theorem 5.15 and Remark 5.5 that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx = \inf_P U(f,P) = \alpha(b-a).$$

5.2 Riemann Sums

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5.17 Definition.

Let $f: [a,b] \to \mathbb{R}$.

1. A Riemann sum of f with respect to a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum
$$\mathcal{S}(f, P, t_j) := \sum_{j=1}^{n} f(t_j)\, \Delta x_j\,.$$
2. The Riemann sums of f are said to converge to $I(f)$ as $||P|| \to 0$ if and only if given $\epsilon > 0$ there is a partition P_ϵ of $[a,b]$ such that
$$P = \{x_0, \dots, x_n\} \supseteq P_\epsilon \text{ implies } \left|\mathcal{S}(f, P, t_j) - I(f)\right| < \epsilon$$
For all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1,2, \dots, n$. In this case we shall use the notation
$$I(f) = \lim_{||P|| \to 0} \mathcal{S}(f, P, t_j) := \lim_{||P|| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j\,.$$

5.18 Theorem.

Let $a, b \in \mathbb{R}$ with $a < b$, and suppose that $f: [a,b] \to \mathbb{R}$. Then f is Riemann integrable on $[a,b]$ if and only if
$$I(f) = \lim_{||P|| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j$$
Exists, in which case $I(f) = \int_a^b f(x)\, dx$.

Proof.

Suppose that f is integrable on $[a,b]$ and that $\epsilon > 0$. By the Approximation Property, there is a partition P_ϵ of $[a,b]$ such that
$$L(f, P_\epsilon) > \int_a^b f(x)\, dx - \epsilon \quad \text{and} \quad U(f, P_\epsilon) < \int_a^b f(x)\, dx + \epsilon.$$
Let $P = \{x_0, x_1, \dots, x_n\} \supseteq P_\epsilon$. Then (4) holds with P in place of P_ϵ. But $m_j(f) \le f(t_j) M = M_j(f)$ for any choice of $t_j \in [x_{j-1}, x_j]$. Hence,
$$\int_a^b f(x)\, dx - \epsilon < L(f, P) \le \sum_{j=1}^{n} f(t_j)\, \Delta x_j \le U(f, P) < \int_a^b f(x)\, dx + \epsilon$$
That is, $-\epsilon < \sum_{j=1}^{n} f(y_j)\, \Delta x_j - \int_a^b f(x)\, dx < \epsilon$. We conclude that
$$\left|\sum_{j=1}^{n} f(t_j)\, \Delta x_j - \int_a^b f(x)\, dx\right| < \epsilon$$
For all partitions $P \supseteq P_\epsilon$ and all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1,2, \dots, n$.

Conversely, suppose that the Riemann sums of f converge to $I(f)$. Let $\epsilon > 0$ and choose a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that
$$\left|\sum_{j=1}^{n} f(t_j)\, \Delta x_j - I(f)\right| < \frac{\epsilon}{3}$$
For all choices of $t_j \in [x_{j-1}, x_j]$. Since f is bounded on $[a,b]$ (see Exercise 5.2.11), use the Approximation Property to choose $t_j, u_j \in [x_{j-1}, x_j]$ such that $f(t_j) - f(u_j) > M_j(f) -$

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{j=1}^{n} \left(M_j(f) - m_j(f) \right) \Delta x_j \\
&< \sum_{j=1}^{n} \left(f\left(t_j\right) - f\left(u_j\right) \right) \Delta x_j + \frac{\epsilon}{3(b-a)} \sum_{j=1}^{n} \Delta x_j \\
&\le \left| \sum_{j=1}^{n} f\left(t_j\right) \Delta x_j - I(f) \right| + \left| I(f) - \sum_{j=1}^{n} f\left(u_j\right) \Delta x_j \right| + \frac{\epsilon}{3(b-a)} \sum_{j=1}^{n} \Delta x_j \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

Therefore, f in integrable on $[a,b]$.

5.19 Theorem. [Linear Property].

If f, g are integrable on $[a,b]$ and $\alpha \in \mathbb{R}$, then $f+g$ and αf are integrable on $[a,b]$ are integrable on $[a,b]$. In fact,

$$\int_a^b \left(f(x) + g(x)\right) dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

And

$$\int_a^b \left(\alpha f(x)\right) dx = \alpha \int_a^b f(x)\, dx.$$

Proof.

Let $\epsilon > 0$ and choose P_ϵ such that for any partition $P = \{x_0, x_1, \dots, x_n\} \supseteq P_\epsilon$ of $[a,b]$ and any choice of $t_j \in \left[x_{j-1}, x_j\right]$, we have

$$\left| \sum_{j=1}^{n} f\left(t_j\right) \Delta x_j - \int_a^b f(x)\, dx \right| < \frac{\epsilon}{2}$$

And

$$\left| \sum_{j=1}^{n} g\left(t_j\right) \Delta x_j - \int_a^b g(x)\, dx \right| < \frac{\epsilon}{2}$$

By the Triangle Inequality,

$$\left| \sum_{j=1}^{n} f\left(t_j\right) \Delta x_j + \sum_{j=1}^{n} g\left(t_j\right) \Delta x_j - \int_a^b f(x)\, dx - \int_a^b g(x)\, dx \right| < \epsilon$$

For any choice of $t_j \in \left[x_{j-1}, x_j\right]$. Hence, (6) follows directly from Theorem 5.18.

To prove (7), we may suppose that $\alpha \neq 0$. Choose P_ϵ such that if $P = \{x_0, \dots, x_n\}$ is finer than P_ϵ, then

$$\left| \sum_{j=1}^{n} f\left(t_j\right) \Delta x_j - \int_a^b f(x)\, dx \right| < \frac{\epsilon}{|\alpha|}$$

For any choice of $t_j \in \left[x_{j-1}, x_j\right]$. Multiplying this inequality by $|\alpha|$, we obtain

$$\left| \sum_{j=1}^{n} \alpha f\left(t_j\right) \Delta x_j - \alpha \int_a^b f(x)\, dx \right| < |\alpha| \frac{\epsilon}{|\alpha|} = \epsilon$$

For any choice of $t_j \in \left[x_{j-1}, x_j\right]$. We conclude by Theorem 5.18 that (7) holds.

5.20 Theorem.

If f is integrable on $[a,b]$, then f is integrable on each subinterval $[c,d]$ of $[a,b]$. Moreover,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

For all $c \in (a,b)$.

Proof.
We may suppose that $a < b$. Let $\epsilon > 0$ and choose a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Let $P' = P \cup \{c\}$ and $P_1 = P' \cap [a, c]$. Since P_1 is a partition of $[a, c]$ and P' is a refinement of P, we have by (9) that

$$U(f, P_1) - L(f, P_1) \le U(f, P') - L(f, P') \le U(f, P) - L(f, P) < \epsilon$$

Therefore, f is integrable on $[a, c]$. A similar argument proves that f is integrable on any subinterval $[c, d]$ of $[a, b]$.
To verify (8), suppose that P is any partition of $[a, b]$. Let $P_0 = P \cup \{c\}$, $P_1 = P_0 \cap [a, c]$, and $P_2 = P_0 \cap [c, b]$. Then $P_0 = P_1 \cup P_2$ and by definition

$$U(f, P) \ge U(f, P) = U(f, P_1) + U(f, P_2)$$
$$\ge (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(This last equality follows from the fact that f is integrable on both $[a, c]$ and $[c, b]$.) Taking the infimum of

$$U(f, P) \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Over all partitions P of $[a, b]$, we obtain

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx \ge \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

A similar argument using lower integrals shows that

$$\int_a^b f(x)\,dx \le \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Using the conventions

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad \text{and} \quad \int_a^a f(x)\,dx = 0$$

It is easy to see that (8) holds whether or not c lies between a and b, provided f is integrable on the union of these intervals (see Exercise 5.2.4).

5.21 Theorem. [Comparison Theorem for Integrals].
If f, g are integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

In particular, if $m \le f(x) \le M$ for $x \in [a, b]$, then

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Proof.
Let P be a partition of $[a, b]$. By hypothesis, $M_j(f) \le M_j(g)$ whence $U(f, P) \le U(g, P)$. It follows that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx \le U(g, P)$$

For all partitions P of $[a, b]$. Taking the infimum of this inequality over all partitions P of $[a, b]$, we obtain

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

If $m \le f(x) \le M$, them (by what we just proved and by Theorem 5.16)

$$m(b - a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b - a).$$

We shall use the following result nearly every time we need to estimate an integral.

5.22 Theorem.
If f is (Riemann) integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof.

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. We claim that

$$M_j(|f|) - m_j(|f|) \le M_j(f) - m_j(f)$$

Holds for $j = 1,2,\dots,n$. Indeed, let $x, y \in [x_{j-1}, x_j]$. If $f(x), f(y)$ have the same sign, say both are nonnegative, then

$$|f(x)| - |f(y)| = f(x) - f(y) \le M_j(f) - m_j(f).$$

If $f(x), f(y)$ have opposite signs, say $f(x) \ge 0 \ge f(y)$, then $m_j(f) \le 0$ and hence,

$$|f(x)| - |f(y)| = f(x) + f(y) \le M_j(f) + 0 \le M_j(f) - m_j(f).$$

Thus in either case, $|f(x)| \le M_j(f) - m_j(f) + |f(y)|$. Taking the supremum of this last inequality for $x \in [x_{j-1}, x_j]$ and then the infimum as $y \in [x_{j-1}, x_j]$, we see that (10) holds, as promised.

Let $\epsilon > 0$ and choose a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. Since (10) implies $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$, it follows that

$$U(|f|,P) - L(|f|,P) < \epsilon.$$

Thus $|f|$ is integrable on $[a,b]$. Since $-|f(x)| \le f(x) \le |f(x)|$ holds for any $x \in [a,b]$, we conclude by Theorem 5.21 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

5.23 Corollary.

If f and g are (Riemann) integrable on $[a,b]$, then so is fg.

Proof.

Suppose for a moment that the square of any integrable function is integrable. Then, by hypothesis, f^2, g^2, and $(f+g)^2$ are integrable on $[a,b]$. Since

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2},$$

It follows from Theorem 5.19 that fg is integrable on $[a,b]$.

It remains to prove that f^2 is integrable on $[a,b]$. Let P be a partition of $[a,b]$. Since $M_j(f^2) = \left(M_j(|f|)\right)^2$ and $m_j(f^2) = \left(m_j(|f|)\right)^2$, it is clear that

$$\begin{aligned} M_j(f^2) - m_j(f^2) &= \left(M_j(|f|)\right)^2 - \left(m_j(|f|)\right)^2 \\ &= \left(M_j(|f|) + m_j(|f|)\right)\left(M_j(|f|) - m_j(|f|)\right) \\ &\le 2M\left(M_j(|f|) - m_j(|f|)\right) \end{aligned}$$

Where $M = \sup|f|([a,b])$; that is, $|f(x)| \le M$ for all $x \in [a,b]$. Multiplying the displayed inequality by Δx_j and summing over $j = 1,2,\dots,n$, we have

$$U(f^2,P) - L(f^2,P) \le 2M\left(U(|f|,P) - L(|f|,P)\right).$$

Hence, it follows from Theorem 5.22 that f^2 is integrable on $[a,b]$.

5.24 Theorem. [First Mean Value Theorem For Integrals].

Suppose that f and g are integrable on $[a,b]$ with $g(x) \ge 0$ for all $x \in [a,b]$. If

$$m = \inf f[a,b] \quad \text{and } M = \sup f[a,b],$$

Then there is a number $c \in [m,M]$ such that

$$\int_a^b f(x)g(x)\,dx = c\int_a^b g(x)\,dx.$$

In particular, if f is continuous on $[a,b]$, then there is an $x_0 \in [a,b]$ which satisfies

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)\,dx.$$

Proof.

Since $g \geq 0$ on $[a.b]$, Theorem 5.21 implies

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

If $\int_a^b g(x)\,dx = 0$, then $\int_a^b f(x)g(x)\,dx = 0$ and there is nothing to prove.

Otherwise, set

$$c = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$$

And note that $c \in [m, M]$. If f is continuous, then (by the Intermediate Value Theorem) we can choose $x_0 \in [a, b]$ such that $f(x_0) = c$.

5.25 Example.

Find $F(x) = \int_0^x f(t)\,dt$ if

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}.$$

Solution.

By Theorem 5.16,

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Hence, $F(x) = |x|$.

Notice in Example 5.25 that the integral F of f is continuous even though f itself is not. The following result shows that this is a general principle.

5.26 Theorem.

If f is (Riemann) integrable on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ exists and is continuous on $[a, b]$.

Proof.

By Theorem 5.20, $F(x)$ exists for all $x \in [a, b]$. To prove that F is continuous on $[a, b]$, it suffices to show that $F(x+) = F(x)$ for all $x \in [a, b)$ and $F(x-) = F(x)$ for all $x \in (a, b]$. Fix $x_0 \in [a, b)$. By definition, f is bounded on $[a, b]$. Thus, choose $M \in \mathbb{R}$ such that $|f(t)| \leq M$ for all $t \in [a, b]$. Let $\epsilon > 0$ and set $\delta = \frac{\epsilon}{M}$. If $0 \leq x - x_0 < \delta$, then by Theorem 5.22,

$$|F(x) - F(x_0)| = \left|\int_{x_0}^x f(t)\,dt\right| \leq \int_{x_0}^x |f(t)|\,dt \leq M|x - x_0| < \epsilon.$$

Hence, $F(x_0+) = F(x_0)$. A similar argument shows that $F(x_0-) = F(x_0)$ for all $x_0 \in (a, b]$.

5.27 Theorem. [Second Mean Value Theorem For Integrals].

Suppose that f, g are integrable on $[a, b]$, that g is nonnegative on $[a, b]$, and that m, M are real numbers which satisfy $m \leq \inf f([a, b])$ and $M \geq \sup f([a, b])$. Then there is an $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = m\int_a^c g(x)\,dx + M\int_c^b g(x)\,dx.$$

In particular, if f is also nonnegative on $[a, b]$, then there is an $c \in [a, b]$ which satisfies

$$\int_a^b f(x)g(x)\,dx = M\int_c^b g(x)\,dx.$$

Proof.

The second statement follows from the first since we may use $m = 0$ when $f \geq 0$. To prove the first statement, set

$$F(x) = m\int_a^x g(t)\,dt + M\int_x^b g(t)\,dt$$

For $x \in [a, b]$, and observe by Theorem 5.26 that F is continuous on $[a, b]$. Since g is nonnegative, we also have $mg(t) \leq f(t)g(t) \leq Mg(t)$ for all $t \in [a, b]$. Hence, it follows from the Comparison Theorem (Theorem 5.21) that

$$F(b) = m\int_a^b g(t)\,dt \leq \int_a^b f(t)g(t)\,dt \leq M\int_a^b g(t)\,dt = F(a).$$

Since F is continuous, we conclude by the Intermediate Value Theorem that there is an $c \in [a, b]$ such that

$$F(c) = \int_a^b f(t)g(t)\,dt.$$

5.3 The Fundamental Theorem of Calculus

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5.28 Theorem. [Fundamental Theorem of Calculus].
Let $[a, b]$ be nondegenerate and suppose that $f: [a, b] \to \mathbb{R}$.

1. If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F \in \mathcal{C}^1[a, b]$ and
$$\frac{d}{dx}\int_a^x f(t)\,dt := F'(x) = f(x)$$
For each $x \in [a, b]$.
2. If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then
$$\int_a^x f'(t)\,dt = f(x) - f(a)$$
For each $x \in [a, b]$.

Proof.

1. For $x \in [a, b]$, set $F(x) = \int_a^x f(t)\,dt$. By symmetry, it suffices to show that if $f(x_0+) = f(x_0)$ for some $x_0 \in [a, b)$, then
$$\lim_{h\to 0+}\frac{F(x_0+h) - F(x_0)}{h} = f(x_0)$$
(see Definition 4.6). Let $\epsilon > 0$ and choose a $\delta > 0$ such that $x_0 \le t < x_0 + \delta$ implies $|f(t) - f(x_0)| < \epsilon$. Fix $0 < h < \delta$. Notice that by Theorem 5.20,
$$F(x_0+h) - F(x_0) = \int_{x_0}^{x_0+h} f(t)\,dt$$
And that by Theorem 5.16,
$$f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt.$$
Therefore,
$$\frac{F(x_0+h) - F(x_0)}{h} - f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h}\big(f(t) - f(x_0)\big)\,dt.$$
Since $0 < h < \delta$, it follows from Theorem 5.22 and the choice of δ that
$$\left|\frac{F(x_0+h) - F(x_0)}{h} - f(x_0)\right| = \frac{1}{h}\int_{x_0}^{x_0+h}\left|\big(f(t) - f(x_0)\big)\right|\,dt \le \epsilon.$$
This verifies (11) and the proof of part 1) is complete.

2. We may suppose that $x = b$. Let $\epsilon > 0$. Since f' is integrable, choose a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that
$$\left|\sum_{j=1}^{n} f'(t_j)\,\Delta x_j - \int_a^b f'(t)\,dt\right| < \epsilon$$
For any choice of points $t_j \in [x_{j-1}, x_j]$. Use the Mean Value Theorem to choose points $t_j \in [x_{j-1}, x_j]$ such that $f(x_j) - f(x_{j-1}) = f'(t_j)\,\Delta x_j$. It follows by telescoping that
$$\left|f(b) - f(a) - \int_a^b f'(t)\,dt\right| = \left|\sum_{j=1}^{n}\big(f(x_j) - f(x_{j-1})\big) - \int_a^b f'(t)\,dt\right| < \epsilon.$$

5.29 Remark.
The hypotheses of the Fundamental Theorem of Calculus cannot be relaxed.

Proof.

1. Define f on $[-1,1]$ by
$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$
Then f is integrable on $[-1,1]$, but $F(x) \coloneqq \int_{-1}^{x} f(x)\,dx = |x| - 1$ is not differentiable at $x = 0$.
2. Define f on $[0,1]$ by $f(x) \coloneqq x^2 \sin\left(\frac{1}{x^2}\right)$ when $x \neq 0$ and $f(0) = 0$. Then f is differentiable on $[0,1]$, but
$$f'(x) = 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cdot \cos\left(\frac{1}{x^2}\right). \quad x \neq 0$$
Is not even bounded on $(0,1]$, much less integrable on $[0,1]$.

By the Fundamental Theorem of Calculus, integration is the inverse of differentiation in the following sense. If f' is integrable, then
$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b \coloneqq f(b) - f(a).$$
In particular,
$$\int_a^b x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1}\Bigg|_a^b$$
For each $\alpha \geq 0$, and for each $\alpha < 0$, provided $\alpha \neq -1$ and $[a,b]$ is a subset of $(0,\infty)$ (see Exercises 4.2.3 and 5.3.7). This result is sometimes called the *Power Rule*.

5.30 Examples.
1. Find $\int_0^1 (3x-2)^2\,dx$.
2. Find $\int_0^{\frac{\pi}{2}} (1 + \sin x)\,dx$.

Solution.
1. Since $(3x-2)^2 = 9x^2 - 12x + 4$, we have by the Power Rule that
$$\int_0^1 (3x-2)^2\,dx = 3x^3 - 6x^2 + 4x\Big|_0^1 = 1.$$
2. Since $(\cos x)' = -\sin x$, we have by the Fundamental Theorem of Calculus that
$$\int_0^{\frac{\pi}{2}} (1 + \sin x)\,dx = x - \cos x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} + 1.$$

5.31 Theorem. [Integration by Parts].
Suppose that f, g are differentiable on $[a,b]$ with f', g' integrable on $[a,b]$. Then
$$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx.$$
This rule is sometimes abbreviated as
$$\int u\,dv = uv - \int v\,du.$$
Where it is understood that if $w = h(x)$ for some differentiable function h, then the Leibnizian differential dw is defined by $dw = h'(x)dx$.
Integration by parts can be used to reduce the exponent n on an expression of the form $(ax + b)^n f(x)$ when f is integrable.

5.32 Example.
Find $\int_0^{\frac{\pi}{2}} x \sin x\,dx$.

Solution.
Let $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$. Hence, by parts,
$$\int_0^{\frac{\pi}{2}} x \sin x\,dx = -x\cos x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)\,dx = \sin x\Big|_0^{\frac{\pi}{2}} = 1.$$

5.33 Example.
Find $\int_1^3 \log x\, dx$.

Solution.
Let $u = \log x$ and $dv = dx$. Then $du = \frac{dx}{x}$ and $v = x$. Hence, by parts

$$\int_1^3 \log x\, dx = x \log x \Big|_1^3 - \int_1^3 dx = 3 \log 3 - 2.$$

5.34 Theorem. [Change of Variables].
Let ϕ be continuously differentiable on a closed, nondegenerate interval $[a, b]$. If

ϕ' is nonzero on $[a, b]$,

And if

f is integrable on $[c, d] := \phi[a, b]$,

Then $f \circ \phi \cdot |\phi'|$ is integrable on $[a, b]$, and

$$\int_c^d f(t)\, dt = \int_a^b f(\phi(x)) \cdot |\phi'(x)|\, dx.$$

Strategy: By the Mean Value Theorem, hypothesis (12) implies that ϕ is 1-1 on $[a, b]$. Hence by Theorem 4.32, ϕ is strictly monotone on $[a, b]$ and $[c, d] := \phi[a, b]$ is a closed interval. Suppose that ϕ is strictly increasing on $[a, b]$; that is, $|\phi'| = \phi'$ and $[c, d] = [\phi(a), \phi(b)]$. By Theorem 4.32, ϕ^{-1} is increasing on $[c, d]$. Thus if $P = \{t_0, t_1, \dots, t_n\}$ is a partition of $[c, d]$ and $x_j = \phi^{-1}(t_j)$, then $P := \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$. A Riemann sum of the right side of (14) looks like

$$\mathcal{S}\left(f \circ \phi \cdot |\phi'|, P, s_j\right) = \sum_{j=1}^n f\left(\phi\left(s_j\right)\right) \left|\phi'\left(s_j\right)\right| \Delta t_j.$$

On the other hand, a typical term of a Riemann sum of the left side of (14) looks like

$$f\left(u_j\right) \Delta x_j = f\left(u_j\right)\left(\phi\left(t_j\right) - \phi\left(t_{j-1}\right)\right).$$

Since ϕ', hence ϕ, is continuous, we can use the Intermediate Value Theorem to choose $s_j \in [x_{j-1}, x_j]$ such that $u_j = \phi\left(s_j\right)$, and the Mean Value Theorem to choose $c_j \in [x_{j-1}, x_j]$ such that $\phi\left(x_j\right) - \phi\left(x_{j-1}\right) = \phi'\left(c_j\right) \Delta x_j$. It follows that a Riemann sum the left side of (14) looks like

$$\mathcal{S}\left(f, \tilde{P}, u_j\right) = \sum_{j=1}^n f\left(\phi\left(s_j\right)\right) \phi'\left(c_j\right) \Delta x_j.$$

The only difference between this last sum and (15) is that s_j has been replaced by c_j. Since c_j and s_j both belong to the interval $[x_{j-1}, x_j]$ and ϕ' is continuous, making this replacement should not change S much if the norm of P is small enough. Hence, a Riemann sum of the left side of (14) is approximately equal to a Riemann sum of the right side of (14). This means the integrals in (14) should be equal. Here are the details.

Case 1. Suppose that ϕ is strictly increasing on $[a, b]$. Let $\epsilon > 0$. Since f is bounded, choose $M \in (0, \infty)$ such that $|f(x)| \le M$ for all $x \in [c, d]$. Since ϕ' is uniformly continuous on $[a, b]$, choose $\delta > 0$ such that

$$\left|\phi'\left(s_j\right) - \phi'\left(c_j\right)\right| < \frac{\epsilon}{2M(b-a)};$$

That is,

$$\left|f\left(\phi\left(s_j\right)\right)\left(\phi'\left(s_j\right) - \phi'\left(c_j\right)\right)\right| < \frac{\epsilon}{2(b-a)}$$

For all $s_j, c_j \in [a, b]$ with $\left|s_j - c_j\right| < \delta$.
Next, use the Inverse Function Theorem to verify that ϕ^{-1} is continuously differentiable on $[c, d]$. Thus there is an $\eta > 0$ such that if $s, c \in [c, d]$ and $|s - c| < \eta$, then $|\phi^{-1}(s) - \phi^{-1}(c)| < \delta$.
Finally, since f is integrable on $[c, d] = [\phi(a), \phi(b)]$, choose a partition $P = \{t_0, t_1, \dots, t_n\}$ of $[c, d]$ such that $||P|| < \eta$ and

$$\left| \mathcal{S}\left(f, P, u_j\right) - \int_{\phi(a)}^{\phi(b)} f(t)\,dt \right| < \frac{\epsilon}{2}$$

Holds for any choice of $u_j \in \left[t_{j-1}, t_j\right]$.

Set $x_j = \phi^{-1}(t_j)$ and observe (by the choice of η) that $\tilde{P} \coloneqq \left\{x_0, \ldots, x_n\right\}$ is a partition of $[a, b]$ which satisfies $\left\|\tilde{P}\right\| < \delta$.

Let $s_j \in \left[x_{j-1}, x_j\right]$, set $u_j = \phi\left(s_j\right)$, and apply the Mean Value Theorem to choose $c_j \in \left[x_{j-1}, x_j\right]$ such that $\phi\left(x_j\right) - \phi\left(x_{j-1}\right) = \phi'\left(c_j\right)\Delta x_j$. Then, by the choices of c_j, u_j, and t_j, we have $u_j \in \left[t_{j-1}, t_j\right]$ and

$$f\left(\phi\left(s_j\right)\right)\phi'\left(c_j\right)\Delta x_j = f\left(u_j\right)\left(\phi\left(x_j\right) - \phi\left(x_{j-1}\right)\right) = f\left(u_j\right)\left(t_j - t_{j-1}\right).$$

Hence, it follows from (16) and (17) that

$$\begin{aligned}\left|\sum_{j=1}^{n} f\left(\phi\left(s_j\right)\right)\phi'\left(s_j\right)\Delta x_j - \int_{\phi(a)}^{\phi(b)} f(t)\,dt\right| &\leq \left|\sum_{j=1}^{n} f\left(\phi\left(s_j\right)\right)\left(\phi'\left(s_j\right) - \phi'\left(c_j\right)\right)\Delta x_j\right| \\ &\quad + \left|\sum_{j=1}^{n} f\left(u_j\right)\left(t_j - t_{j-1}\right) - \int_{\phi(a)}^{\phi(b)} f(t)\,dt\right| \\ &< \frac{\epsilon}{2(b-a)}\sum_{j=1}^{n}\Delta x_j + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

We obtained this estimate for the fixed partition $\tilde{P}$ of $[a, b]$, but the same steps also verify this estimate for any partition finer than $\tilde{P}$. We conclude by Theorem 5.18 that $\left(f \circ \phi\right) \cdot \left|\phi'\right|$ is integrable on $[a, b]$ and (14) Holds.

Case 2.

ϕ is strictly decreasing on $[a, b]$. Repeat the proof in case 1. The only changes are $\tilde{P} = \left\{\phi^{-1}\left(t_n\right), \ldots, \phi^{-1}\left(t_0\right)\right\}$ and $\left|\phi'\right| = -\phi'$. Thus the Mean Value Theorem implies that

$$\phi\left(x_{j-1}\right) - \phi\left(x_j\right) = \phi'\left(c_j\right)\left(c_{j-1} - x_j\right) = \left|\phi'\left(c_j\right)\right|\Delta x_j.$$

Estimating the Riemann sums as above, we again conclude that

$$\int_c^d f(t)\,dt = \int_a^b f\left(\phi(x)\right) \cdot \left|\phi'(x)\right|\,dx.$$

The proof of Theorem 5.34 also establishes the following more familiar form of the Change of Variables Formula: If ϕ is $\mathcal{C}^1$ on $[a, b]$, if ϕ' is never zero on $[a, b]$, and if f is integrable on $\phi[a, b]$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f\left(\phi(x)\right)\phi'(x)\,dx.$$

The difficult part of Theorem 5.34 was verifying that $f \circ \phi \cdot |\phi'|$ is integrable on $[a, b]$ when f is integrable on $[c, d]$. If we assume that f is continuous, the proof is a lot easier.

5.35 Theorem. [Change of Variables for Continuous Integrands].

If ϕ is continuously differentiable on a closed, nondegenerate interval $[a, b]$ and f is continuous on $\phi([a, b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f\left(\phi(x)\right)\phi'(x)\,dx$$

Proof.

Set

$$G(x) \coloneqq \int_a^x f\left(\phi(t)\right)\phi'(t)\,dt, \qquad x \in [a, b], \qquad \text{and}$$

$$F(u) \coloneqq \int_{\phi(a)}^{u} f(t)\,dt, \qquad u \in \phi([a, b]),$$

And observe that if m is the infimum of $\phi([a,b])$, then $F(u) = \int_m^u f(t)\,dt - \int_m^{\phi(a)} f(t)\,dt$. It follows from the Fundamental Theorem of Calculus that $G'(x) = f(\phi(x))\phi'(x)$ and $F'(u) = f(u)$. Hence, by the Chain Rule,

$$\frac{d}{dx}\Big(G(x) - F\big(\phi(x)\big)\Big) = 0$$

For all $x \in [a,b]$. It follows from Theorem 4.17ii that $G(x) - F\big(\phi(x)\big)$ is constant on $[a,b]$. Evaluation at $x = a$ shows that this constant is zero. Thus $G(x) = F\big(\phi(x)\big)$ for all $x \in [a,b]$, in particular, when $x = b$.

These Change of Variables Formulas can be remembered as a substitution if we use the Leibnizian differentials introduced above: $u = \phi(x)$ implies $du = \phi'(x)dx$.

5.36 Example.
Suppose that f is an unknown function which is nonnegative and continuous on $[2,5]$. If data are collected that can be interpreted as $\int_2^5 f(x)\,dx = 3$, find an upper bound for the integral

$$I := \int_1^2 f\big(x^2+1\big)\,dx.$$

Solution.
Let $u = x^2 + 1$. Then $du = 2x\,dx$. Unlike textbook-style problems, we do not have a du term already in I. However, since $x \in [1,2]$ implies $x \geq 1$, and since f is nonnegative, it is clear that $f\big(x^2+1\big) \leq \frac{2xf(x^2+1)}{2}$. Therefore,

$$I = \int_1^2 f(x^2+1)\,dx \leq \frac{1}{2}\int_1^2 2xf\big(x^2+1\big)\,dx = \frac{1}{2}\int_2^5 f(u)\,du = \frac{3}{2}.$$

5.4 Improper Riemann Integration

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5.37 Remark.
If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,dx\right).$$

Proof.
By Theorem 5.26,

$$F(x) = \int_a^x f(t)\,dt$$

Is continuous on $[a, b]$. Thus

$$\int_a^b f(x)\,dx = F(b) - F(a) = \lim_{c\to a+}\left(\lim_{d\to b-}\big(F(d) - F(c)\big)\right)$$
$$= \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,dx\right).$$

This leads to the following generalization of the Riemann integral.

5.38 Definition.
Let (a, b) be a nonempty, open (possibly unbounded) interval and $f\colon (a, b) \to \mathbb{R}$.

1. f is said to be *locally integrable* on (a, b) if and only if f is integrable on each closed subinterval $[c, d]$ of (a, b).
2. f is said to be *improperly integrable* on (a, b) if and only if f is locally integrable on (a, b) and
$$\int_a^b f(x)\,dx \coloneqq \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,dx\right)$$
Exists and is finite. This limit is called the *improper* (*Riemann*) integral of f over (a, b).

5.39 Remark.
The order of the limits in (18) does not matter. In particular, if the limit in (18) exists, then

$$\int_a^b f(x)\,dx = \lim_{d\to b-}\left(\lim_{c\to a+}\int_c^d f(x)\,dx\right).$$

Proof.
Let $x_0 \in (a, b)$ be fixed. By Theorem 5.20 and 3.8,

$$\lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(x)\,dx\right) = \lim_{c\to a+}\left(\int_c^{x_0} f(x)\,dx + \lim_{d\to b-}\int_{x_0}^d f(x)\,dx\right)$$
$$= \lim_{c\to a+}\int_c^{x_0} f(x)\,dx + \lim_{d\to b-}\int_{x_0}^d f(x)\,dx$$
$$= \lim_{d\to b-}\left(\lim_{c\to a+}\int_c^d f(x)\,dx\right).$$

Thus we shall use the notation

$$\lim_{\substack{c\to a+\\ d\to b-}}\int_c^d f(x)\,dx$$

To represent the limit in (18). If the integral is not improper at one of the endpoints - for example, if f is Riemann integrable on closed subintervals of $(a, b]$ - we shall say that f is improperly integrable on $(a, b]$ and simplify the notation even further by writing

$$\int_a^b f(x)\,dx = \lim_{c\to a+}\int_c^b f(x)\,dx\,.$$
The following example shows that an improperly integrable function need not be bounded.

5.40 Example.
Show that $f(x) = \frac{1}{\sqrt{x}}$ is improperly integrable on $(0,1]$.

Solution.
By definition,
$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0+}\int_a^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0+}\left(2 - 2\sqrt{a}\right) = 2.$$

5.41 Example.
Show that $f(x) = \frac{1}{x^2}$ is improperly integrable on $[1,\infty)$.

Solution.
By definition,
$$\int_1^\infty \frac{1}{x^2}dx = \lim_{d\to\infty}\int_1^d \frac{1}{x^2}dx = \lim_{d\to\infty}\left(1-\frac{1}{d}\right) = 1.$$

5.42 Theorem.
If f, g are improperly integrable on (a,b) and $\alpha,\beta \in \mathbb{R}$, then $\alpha f + \beta g$ is improperly integrable on (a,b) and
$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx.$$
Proof.
By Theorem 5.19 (the Linear Property for Riemann Integrals),
$$\int_c^d \big(\alpha f(x) + \beta g(x)\big)\,dx = \alpha\int_c^d f(x)\,dx + \beta\int_c^d g(x)\,dx$$
For all $a < c < d < b$. Taking the limit as $c \to a+$ and $d \to b-$ finishes the proof.

5.43 Theorem. [Comparison Theorem for Improper Integrals].
Suppose that f, g are locally integrable on (a,b). If $0 \le f(x) \le g(x)$ for $x \in (a,b)$, and g is improperly integrable on (a,b), then f is improperly integrable on (a,b) and
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof.
Fix $c \in (a,b)$. Let $F(d) = \int_c^d f(x)\,dx$ and $G(d) = \int_c^d g(x)\,dx$ for $d \in [c,b)$. By the Comparison Theorem for Integrals, $F(d) \le G(d)$. Since $f \ge 0$, the function F is increasing on $[c,b]$; hence $F(b-)$ exists (see Theorem 4.18). Thus, by definition, f is improperly integrable on (c,b) and
$$\int_c^b f(x)\,dx = F(b-) \le G(b-) = \int_c^b g(x)\,dx.$$
A similar argument works for the case $c \to a+$.

5.44 Example.
Prove that $f(x) = \frac{\sin x}{\sqrt{x^3}}$ is improperly integrable on $(0,1]$.

Proof.
Since f is continuous on $(0,1]$, f is locally integrable there as well. Since f is nonnegative on $(0,1]$, it is clear that $0 \le f(x) = \left|\frac{\sin x}{\sqrt{x^3}}\right| \le \frac{|x|}{x^{\frac{3}{2}}} = \frac{1}{\sqrt{x}}$ on $(0,1]$. Since this last function is improperly integrable on $(0,1]$ by Example 5.40, it follows from the Comparison Test that $f(x)$ is

improperly integrable on (0,1].

5.45 Example.
Prove that $f(x) = \frac{\log x}{\sqrt{x^5}}$ is improperly integrable on $[1, \infty)$.

Proof.
Since f is continuous on $(0, \infty)$, f is integrable on $[1, C]$ for any $C \in [1, \infty)$. By Exercise 4.4.6, there is a constant $C > 1$ such that $0 \le f(x) = \frac{\log x}{\sqrt{x^5}} \le \frac{x^{\frac{1}{2}}}{x^{\frac{5}{2}}} = \frac{1}{x^2}$ for $x \ge C$. Since this last function is improperly integrable on $[1, \infty)$ by Example 5.41, it follows from the Comparison Theorem that $f(x)$ is improperly integrable on $[1, \infty)$.

5.46 Remark.
If f is bounded and locally integrable on (a, b) and $|g|$ is improperly integrable on (a, b), then $|fg|$ is improperly integrable on (a, b).

Proof.
Let $M = \sup_{x\in(a,b)}|f(x)|$. Then $0 \le |f(x)g(x)| \le M|g(x)|$ for all $x \in (a, b)$. Hence, by Theorem 5.43, $|fg|$ is improperly integrable on (a, b).

5.47 Definition.
Let (a, b) be a nonempty, open interval and $f: (a, b) \to \mathbb{R}$.

1. f is said to be *absolutely integrable* on (a, b) if and only if f is locally integrable and $|f|$ is improperly integrable on (a, b).
2. f is said to be *conditionally integrable* on (a, b) if and only if f is improperly integrable but not absolutely integrable on (a, b).

5.48 Theorem.
If f is absolutely integrable on (a, b), then f is improperly integrable on (a, b) and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof.
Since $0 \le |f(x)| + f(x) \le 2|f(x)|$, we have by Theorem 5.43 that $|f| + f$ is improperly integrable on $[a, b]$. Hence, by Theorem 5.42, so is $f = (|f| + f) - |f|$. Moreover,

$$\left|\int_c^d f(x)\,dx\right| \le \int_c^d |f(x)|\,dx$$

For every $a < c < d < b$. We finish the proof by taking the limit of this last inequality as $c \to a+$ and $d \to b-$.

The converse of Theorem 5.48, however, is false.

5.49 Example.
Prove that the function $\frac{\sin x}{x}$ is conditionally integrable on $[1, \infty)$.

Proof.
Integrating by parts, we have

$$\int_1^d \frac{\sin x}{x}\,dx = -\frac{\cos x}{x}\Big|_1^d - \int_1^d \frac{\cos x}{x^2}\,dx$$

$$= \cos(1) - \frac{\cos d}{d} - \int_1^d \frac{\cos x}{x^2}\,dx$$

Since $\frac{1}{x^2}$ is absolutely integrable on $[1, \infty)$, it follows from Remark 5.46 that $\frac{\cos x}{x^2}$ is absolutely integrable on $[1,\infty)$. Therefore, $\frac{\sin x}{x}$ is improperly integrable on $[1,\infty)$ and

$$\int_1^\infty \frac{\sin x}{x}\,dx = \cos(1) - \int_1^\infty \frac{\cos x}{x}\,dx.$$

To show that $\frac{\sin x}{x}$ is not absolutely integrable on $[1,\infty)$, notice that

$$\begin{aligned}\int_1^{n\pi} \frac{|\sin x|}{x}\,dx &\geq \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \\ &\geq \sum_{k=2}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\,dx \\ &= \sum_{k=2}^{n} \frac{2}{k\pi} = \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k}\end{aligned}$$

For each $n \in \mathbb{N}$. Since

$$\sum_{k=2}^{n} \frac{1}{k} \geq \sum_{k=2}^{n} \int_k^{k+1} \frac{1}{x}\,dx = \int_2^{n+1} \frac{1}{x}\,dx = \log(n+1) - \log 2 \to \infty$$

As $n \to \infty$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} \int_1^{n\pi} \frac{|\sin x|}{x}\,dx = \infty.$$

Thus, $\frac{\sin x}{x}$ is not absolutely integrable on $[1,\infty)$.

*5.5 Functions of Bounded Variation

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Place Holder

*5.6 Convex Functions

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Place Holder

Chapter 6 Infinite Series of Real Numbers

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6.1 Introduction

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6.1 Definition.
Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series with terms a_k.

1. For each $n \in \mathbb{N}$, the *partial sum* of S *of order* n is defined by
$$s_n := \sum_{k=1}^{n} a_k \,.$$
2. S is said to *converge* if and only if its sequence of partial sums $\{s_n\}$ converges to some $s \in \mathbb{R}$ as $n \to \infty$; that is, if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \epsilon$. In this case we shall write
$$\sum_{k=1}^{\infty} a_k = s$$
And call s the sum, or value, of the series $\sum_{k=1}^{\infty} a_k$.
3. S is said to *diverge* if and only if its sequence of partial sums $\{s_n\}$ does not converge as $n \to \infty$. When s_n diverges to $+\infty$ as $n \to \infty$, we shall also write
$$\sum_{k=1}^{\infty} a_k = \infty.$$

6.2 Example.
Prove that $\sum_{k=1}^{\infty} 2^{-k} = 1$.

Proof.
By induction, we can show that the partial sums $s_n = \sum_{k=1}^{n} \frac{1}{2^k}$ satisfy $s_n = 1 - 2^{-n}$ for $n \in \mathbb{N}$. Thus $s_n \to 1$ as $n \to \infty$.

6.3 Example.
Prove that $\sum_{k=1}^{\infty} (-1)^k$ diverges.

Proof.
The partial sums $s_n = \sum_{k=1}^{n} (-1)^k$ satisfy
$$s_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$
The s_n does not converge as $n \to \infty$.

6.4 Example. [The Harmonic Series].
Prove that the sequence $\frac{1}{k}$ converges but the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to $+\infty$.

Proof.
The sequence $\frac{1}{k}$ converges to zero (by Example 2.2i). On the other hand, by the Comparison Theorem for Integrals,
$$s_n = \sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} \int_{k}^{k+1} \frac{1}{x} dx = \int_{1}^{n+1} \frac{1}{x} dx = \log(n+1)\,.$$
We conclude that $s_n \to \infty$ as $n \to \infty$.

This example shows that the terms of a divergent series may converge. In particular, a series does not converge just because its terms converge. On the other hand, the following result shows that a series cannot converge if its terms do not converge to zero.

6.5 Theorem. [Divergence Test].
Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. If a_k does not converge to zero, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Proof.
Suppose to the contrary that $\sum_{k=1}^{\infty} a_k$ converges to some $s \in \mathbb{R}$. By definition, the sequence of partial sums $s_n \coloneqq \sum_{k=1}^{n} a_k$ converges to s as $n \to \infty$. Therefore, $a_k = s_k - s_{k-1} \to s - s = 0$ as $k \to \infty$, a contradiction.

6.6 Theorem. [Telescopic Series].
If $\{a_k\}$ is a convergent real sequence, then

$$\sum_{k=1}^{\infty} (a_k - a_{k-1}) = a_1 - \lim_{k\to\infty} a_k .$$

Proof.
By telescoping, we have

$$s_n \coloneqq \sum_{k=1}^{n} (a_k - a_{k+1}) = a_1 - a_{n+1}.$$

Hence, $s_n \to a_1 - \lim_{k\to\infty} a_k$ as $n \to \infty$.

6.7 Theorem. [Geometric Series].
Suppose that $x \in \mathbb{R}$, that $N \in \{0,1,\dots\}$, and that 0^0 is interpreted to be 1. Then the series $\sum_{k=N}^{\infty} x^k$ converges if and only if $|x| < 1$, in which case

$$\sum_{k=N}^{\infty} x^k = \frac{x^N}{1-x}.$$

In particular,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad |x| < 1.$$

Proof.
If $|x| \geq 1$, then $\sum_{k=N}^{\infty} x^k$ diverges by the Divergence Test. If $|x| < 1$, then set $s_n = \sum_{k=1}^{n} x^k$ and observe by telescoping that

$$\begin{aligned}(1-x)s_n &= (1-x)(x + x^2 + \cdots + x^n)\\ &= x + x^2 + \cdots + x^n - x^2 - x^3 - \cdots - x^{n+1} = x - x^{n+1}.\end{aligned}$$

Hence,

$$s_n = \frac{x}{1-x} - \frac{x^{n+1}}{1-x}$$

For all $n \in \mathbb{N}$. Since $x^{n+1} \to 0$ as $n \to \infty$ for all $|x| < 1$ (see Example 2.20), we conclude that $s_n \to \frac{x}{1-x}$ as $n \to \infty$.
For general N, we may suppose that $|x| < 1$ and $x \neq 0$. Hence,

$$\sum_{k=N}^{n} x^k = x^N + \cdots + x^n = x^{N-1} \sum_{k=1}^{n-N+1} x^k .$$

Hence, it follows from Definition 6.1 and what we've already proved that

$$\sum_{k=N}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=N}^{n} x^k = \lim_{n\to\infty} x^{N-1} \sum_{k=1}^{n-N+1} x^k = \frac{x^N}{1-x}.$$

6.8 Theorem. [The Cauchy Criterion].
Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$m \geq n \geq N \ \text{ imply } \ \left|\sum_{k=n}^{m} a_k\right| < \epsilon.$$

Proof.
Let s_n represent the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$ and set $s_0 = 0$. By Cauchy's Theorem (Theorem 2.29), s_n converges if and only if given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $m, n \geq N$ imply $|s_m - s_{n-1}| < \epsilon$. Since

$$s_m - s_{n-1} = \sum_{k=n}^{m} a_k$$

For all integers $m \geq n \geq 1$, the proof is complete.

6.9 Corollary.
Let $\{a_k\}$ be a real sequence. Then the infinite series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad \text{implies} \quad \left|\sum_{k=n}^{\infty} a_k\right| < \epsilon.$$

6.10 Theorem.
Let $\{a_k\}$ and $\{b_k\}$ be real sequences. If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_n$ are convergent series, then

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

And

$$\sum_{k=1}^{\infty} (\alpha a_k) = \alpha \sum_{k=1}^{\infty} a_k$$

For any $\alpha \in \mathbb{R}$.

Proof.
Both identities are corollaries of Theorem 2.12; we provide the details only for the first identity. Let s_n represent the partial sums of $\sum_{k=1}^{\infty} a_k$ and t_n represent the partial sums of $\sum_{k=1}^{\infty} b_k$. Since real addition is commutative, we have

$$\sum_{k=1}^{n} (a_k + b_k) = s_n + t_n, \qquad n \in \mathbb{N}.$$

Taking the limit of this identity as $n \to \infty$, we conclude by Theorem 2.12 that

$$\sum_{k=1}^{\infty} (a_k + b_k) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k .$$

6.2 Series with Nonnegative Terms

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6.11 Theorem.
Suppose that $a_k \geq 0$ for large k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if its sequence of partial sums $\{s_n\}$ is bounded; that is, if and only if there exists a finite number $M > 0$ such that

$$\left|\sum_{k=1}^{n} a_k\right| \leq M \text{ for all } n \in \mathbb{N}.$$

Proof.
Set $s_n = \sum_{k=1}^{n} a_k$ for $n \in \mathbb{N}$. If $\sum_{k=1}^{\infty} a_k$ converges, then s_n converges as $n \to \infty$. Since every convergent sequence is bounded (Theorem 2.8), $\sum_{k=1}^{\infty} a_k$ has bounded partial sums.
Conversely, suppose that $|s_n| \leq M$ for $n \in \mathbb{N}$. Recall from Section 2.1 that $a_k \geq 0$ for large k means that there is an $N \in \mathbb{N}$. Recall from Section 2.1 that $a_k \geq 0$ for large k means that there is an $N \in \mathbb{N}$ such that $a_k \geq 0$ for $k \geq N$. It follows that s_n is an increasing sequence when $n \geq N$. Hence by the Monotone Convergence Theorem (Theorem 2.19), s_n converges.

6.12 Theorem. [Integral Test].
Suppose that $f\colon [1,\infty) \to \mathbb{R}$ is positive and decreasing on $[1,\infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if f is improperly integrable on $[1,\infty)$; that is, if and only if

$$\int_1^{\infty} f(x)\,dx < \infty.$$

Proof.
Let $s_n = \sum_{k=1}^{n} f(k)$ and $t_n = \int_1^n f(x)\,dx$ for $n \in \mathbb{N}$. Since f is decreasing, f is locally integrable on $[1,\infty)$ (see Exercise 5.1.8) and $f(k+1) \leq f(x) \leq f(k)$ for all $x \in [k, k+1]$. Hence, by the Comparison Theorem for Integrals,

$$f(k+1) \leq \int_k^{k+1} f(x)\,dx \leq f(k)$$

For $k \in \mathbb{N}$. Summing over $k = 1, \dots, n-1$, we obtain

$$s_n - f(1) = \sum_{k=2}^{n} f(k) \leq \int_1^n f(x)\,dx = t_n \leq \sum_{k=1}^{n-1} f(k) = s_n - f(n)$$

For all $n \geq N$. In particular,

$$f(n) \leq \sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx \leq f(1) \;\text{ for } n \in \mathbb{N}.$$

By (3) it is clear that $\{s_n\}$ is bounded if and only if $\{t_n\}$ is. Since $f(x) \geq 0$ implies that both s_n and t_n are increasing sequences, it follows from the Monotone Convergence Theorem that s_n converges if and only if t_n converges, as $n \to \infty$.

6.13 Corollary. [*p*-Series Test].
The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Converges if and only if $p > 1$.

Proof.
If $p = 1$ or $p \leq 0$, the series diverges. If $p > 0$ and $p \neq 1$, set $f(x) = x^{-p}$ and observe that $f'(x) = -p^{-p-1} < 0$ for all $x \in [1,\infty)$. Hence, f is nonnegative and decreasing on $[1,\infty)$. Since

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{n\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^n = \lim_{n\to\infty} \frac{n^{1-p} - 1}{1-p}$$

Has a finite limit if and only if $1 - p < 0$, it follows from the Integral Test that (4) converges if and only $p > 1$.

6.14 Theorem. [Comparison Test].
Suppose that $0 \leq a_k \leq b_k$ for large k.

1. If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.
2. If $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$.

Proof.
By hypothesis, choose $N \in \mathbb{N}$ so large that $0 \leq a_k \leq b_k$ for $k > N$. Set $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$, $n \in \mathbb{N}$. Then $0 \leq s_n - s_N \leq t_n - t_N$ for all $n \geq N$. Since N is fixed, it follows that s_n is bounded when t_n is, and t_n is unbounded when s_n is. Apply Theorem 6.11 and the proof of the theorem is complete.

6.15 Example.
Determine whether the series

$$\sum_{k=1}^{\infty} \frac{3k}{k^2 + k}\sqrt{\frac{\log k}{k}}$$

Converges or diverges.

Solution.
The kth term of this series can be written by using three factors:

$$\frac{1}{k}\frac{3k}{k+1}\sqrt{\frac{\log k}{k}}.$$

The factor $\frac{3k}{k+1}$ is dominated by 3. Since $\log k \leq \sqrt{k}$ for large k, the factor $\sqrt{\frac{\log k}{k}}$ satisfies

$$\sqrt{\frac{\log k}{k}} \leq \sqrt{\frac{\sqrt{k}}{k}} = \frac{1}{\sqrt[4]{k}}$$

For large k. Therefore, the terms of (5) are dominated by $\frac{3}{k^{\frac{5}{4}}}$. Since $\sum_{k=1}^{\infty} \frac{3}{k^{\frac{5}{4}}}$ converges by the p-Series Test, it follows from the Comparison Test that (5) converges.

6.16 Theorem. [Limit Comparison Test].
Suppose that $a_k \geq 0$, that $b_k > 0$ for large k, and that $L := \lim_{n\to\infty} \frac{a_n}{b_n}$ exists as an extended real number.

1. If $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.
2. If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
3. If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof.

1. If L is finite and nonzero, then there is an $N \in \mathbb{N}$ such that
$$\frac{L}{2} b_k < a_k < \frac{3L}{2} b_k$$
For $k \geq N$. Hence, part 1 follows immediately from the Comparison Test and Theorem 6.10. Similar arguments establish parts 2. and 3. - see Exercise 6.2.6.

6.17 Example.
Let $a_k \to 0$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} \sin|a_k|$ converges if and only if $\sum_{k=1}^{\infty}|a_k|$ converges.

Proof.
By L'Hospital's Rule,

$$\lim_{k\to\infty} \frac{\sin|a_k|}{|a_k|} = \lim_{x\to 0+} \frac{\sin x}{x} = 1.$$

Hence, by the Limit Comparison Test, $\sum_{k=1}^{\infty} \sin|a_k|$ converges if and only if $\sum_{k=1}^{\infty}|a_k|$ converges.

6.3 Absolute Convergence

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6.18 Definition.
Let $S = \sum_{k=1}^{\infty} a_k$